RATIONALITY AND RELATED PROBLEMS

by

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ABSTRACT

In this manuscript we discuss some problems regarding rational varieties. We study how rationality deforms in families giving a complete description for families of threefolds. We introduce a new invariant that can be attached to a rational variety and we study rational varieties for small values of the invariant. We discuss and we give a partial answer to the problem of determine if for every point on a rational variety we can find a global system of parameters.

For my wife and my family.

"A man should look for what is, and not for what he thinks should be."

– Albert Einstein

CONTENTS

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CHAPTER 1

INTRODUCTION

Rationality and unirationality problems have been one of the main topics in algebraic geometry over the years. An algebraic variety X of dimension n is said to be unirational if there exists a dominant rational map $\phi : \mathbb{P}^n \dashrightarrow X$, if the map $\phi : \mathbb{P}^n \dashrightarrow X$ is birational then the algebraic variety X is said to be rational. Both rational and unirational varieties belong to the larger class of rationally connected varieties. A variety X is said to be rationally connected if for two general points x and y of X, we may find a rational curve connecting x and y. This notion was introduced by Campana ([6]) and Kollar-Miyaoka-Mori ([24]).

The starting point is the classic *Lüroth's Theorem* (1876), which states that an algebraic irreducible curve defined over $\mathbb{C}$ is unirational if and only if it is rational. The Lüroth's Theorem came into the following problem: let X be an algebraic variety of arbitrary dimension, if X is unirational then does this imply that X is rational?

The answer is positive in dimension 2, due to the *Rationality Criterium of Castelnuovo* (1892), which states that an algebraic surface defined over $\mathbb{C}$ is rational if and only if $P_2(X) = q(X) = 0$, where $P_r(X) = \dim H^0(K_X^{\otimes 2})$ (plurigenera) is the dimension of the vector space of the r-tensor of holomorphic n-forms defined over X and $q(X) = \dim H^0(\Omega_X^1)$ (irregularity) is the dimension of the vector space of the holomorphic 1-forms defined over X. Since it is well known that if an algebraic variety of arbitrary dimension is unirational then $P_r(X) = q(X) = 0$ for every r, the rationality criterium of Castelnuovo tells us that for surfaces being unirational is equivalent to being rational. The question in arbitrary dimension had been open for one century, and it had an answer with the works of Clemens-Griffiths ([7]) and Iskovskikh-Manin ([20]). In their work Clemens and Griffiths proved that every smooth cubic hypersurface in $\mathbb{P}^4_{\mathbb{C}}$ is unirational but not rational. Iskovskikh and Manin proved that every smooth quartic hypersurface in $\mathbb{P}^4_{\mathbb{C}}$ is not rational. This combined with the examples of Segre ([28]) of smooth quartic hypersurface in $\mathbb{P}^4_{\mathbb{C}}$ that are unirational gives a counterexample to the Lüroth's problem.

In particular, the works of Clemens-Griffiths and Iskovskikh-Manin show that there is no hope of finding a rationality criterium in arbitrary dimension using the classic cohomological invariant of rationality, i.e., irregularity and plurigenera.

The problem of a full understanding of rational, unirational and rational connected varieties of higher dimension is still widely open. For instance, an example of an algebraic variety that is rationally connected but not unirational is at the moment unknown.

The core of this manuscript consists in the study of three problems regarding rational varieties.

The first problem is the description of how rationality deforms in smooth families. The problem is classical and it is contained in the following long-standing open question:

Question 1. *Given a smooth family of complex projective varieties $f : X \to Y$, is the set of rational fibers a countable union of closed subsets of Y?*

The answer is positive in dimension two due to Castelnuovo's Theorem. We give a positive answer to this question in dimension three, and provide an analogous result in arbitrary characteristic. The key result is the following theorem.

Theorem 1. *Let $f : X \to T$ be a projective morphism from a variety X onto a smooth curve T defined over an uncountable algebraically closed field k. Let $0 \in T$ be a closed point. Assume that X_t is a rational variety for every $t \neq 0$, and that the central fiber X_0 has dimension three. Then every irreducible component D of X_0 that is separably rationally connected is rational.*

This implies a positive answer to question 1 for smooth families of threefolds by the following general property.

Proposition 1. *Let $f : X \to Y$ be a morphism of proper projective varieties. Then the set of rational fibers is countable union of locally closed subset of Y.*

The second problem is the classification of rational varieties. Despite the fact that we do not classify all rational varieties, we make an attempt to introduce a new intrinsic invariant of rational varieties, called rational degree, and we classifify rational varieties for small values of the invariant.

It has been observed by Ionescu and Russo, [19], one can characterize rationality as follows: a variety X is rational if and only if there exists a covering family $\mathcal{C} \subset \mathrm{Chow}(X)$

of 1-cycles with rational components through a point $x \in X_{reg}$ with all the cycles smooth at x and such that the general cycle of the family is uniquely determined by its tangent direction at x.

Building on Ionescu-Russo's work, we make this more precise. Namely, we show that the family of rational curves as above determines a birational map $\phi : X \dashrightarrow \mathbb{P}^n$, that restricts to an isomorphism in an open neighborhood of x. Even more, we prove that the total space of the family $\mathcal{C}$ gives a resolution of the birational map ϕ. We define the rational degree of $\mathcal{C}$ on X, shortly $\operatorname{ratdeg}_{\mathcal{C}}(X)$, the intersection number $D \cdot C$, where $[C] \in \mathcal{C}$, and D is a member of the linear system defining ϕ. We call rational degree of X, shortly $\operatorname{ratdeg}(X)$, the minimum of all possible $\operatorname{ratdeg}_{\mathcal{C}}(X)$.

For small values of the rational degree of X we have the following.

Proposition 2. *Let X be a smooth rational variety, with $\dim X = n$ and $\operatorname{ratdeg}(X) = 1$. Then there exists a birational morphism $X \to \mathbb{P}^n$.*

The following result begins the classification of varieties with rational degree two.

Theorem 2. *Let X be a rational variety with $\dim X = 3$ and $\operatorname{ratdeg}(X) = 2$. Then there exists a birational morphism $X \to Y$, where Y is a variety of minimal degree. In particular Y is one of the following:*
i) $Q^3 \subset \mathbb{P}^4$
ii) $S(0,0,2) \subset \mathbb{P}^4$
iii) $S(0,1,1) \subset \mathbb{P}^4$
iv) $S(0,1,2) \subset \mathbb{P}^5$
v) $S(1,1,1) \subset \mathbb{P}^5$
vi) $C(v_2(\mathbb{P}^2))$
where Q^3 is the quadric hypersuface in $\mathbb{P}^4$, $C(v_2(\mathbb{P}^2))$ is the cone over the Veronese surface, and $S(a_0,a_1,a_2)$ is the image in $\mathbb{P}^{\sum a_i+2}$ of $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(a_0) \oplus \mathcal{O}(a_1) \oplus \mathcal{O}(a_2))$ under the tautological map. All the cases are effective.

Corollary 1. *Let X be a smooth rational variety, with $\dim X \geq 3$, $\rho(X) = 1$ and* $\operatorname{ratdeg}(X) = 2$. *Then $X \cong Q^n \subset \mathbb{P}^{n+1}$.*

The third problem arises in a natural way from the dualism of complex varieties that can be studied with techniques both from complex geometry and from algebraic geometry.

Every point p in a n-fold M admits locally, in the euclidean topology, a neighborhood biholomorphic to an open disc in $\mathbb{C}^n$. When we consider Zariski topology, only rational n-folds admit an open neighborhood biholomorphic to an open subset of $\mathbb{C}^n$. It is interesting investigate if every point x on a rational n-fold X admits a system of global parameters centered at x.

The following question was raised by Pandharipande:

Question 2. *If a smooth variety X of dimension n is rational, then is it the case that for every point $x \in X$ there exists an open neighborhood U of x isomorphic to an open set of the projective space $\mathbb{P}^n$?*

It is generally expected that the answer to Question 2 should be negative. In fact an expected counter example was the blow-up of $\mathbb{P}^n$ along a subvariety far away from being rational. The points on the exceptional divisor were expected not to admit an open set isomorphic to an open set of the projective space $\mathbb{P}^n$.

While we have not been able to give a complete answer to Question 2, we prove that the property of admitting a global system of parameters centered at one point has good behavior under blow up. That is:

Proposition 3. *Let X be a smooth rational projective variety, and $x \in X$ be a point that admits an open set isomorphic to an open set of the projective space $\mathbb{P}^n$. If $f : \mathrm{Bl}_Z(X) \to X$ is the blow up of X along a smooth subvariety Z containing the point x, then every point $y \in f^{-1}(x)$ admits an open set isomorphic to an open set of the projective space $\mathbb{P}^n$.*

Proposition 3 shows that if the answer to Question 2 is negative, then counter examples cannot be constructed by just taking blow-ups.

CHAPTER 2

BACKGROUND

We work over an algebraically closed field k. All schemes are assumed to be of finite type over k. With the term *variety* we mean an integral scheme, and we use the term *n-fold* to denote a variety of dimension n.

2.1 General definitions and deformation of rational curves

We recall the basic definitions of rational, unirational and rational connected varieties, we refer to [22] for more details.

Definition 1. *A variety X of dimension n is said to be* rational *if there exists a birational map $\phi : X \dashrightarrow \mathbb{P}^n$.*

Definition 2. *A variety X of dimension n is said* unirational *if there exists a dominant rational map $\phi : \mathbb{P}^N \dashrightarrow X$.*

Definition 3. *A variety X is said* rationally connected *if there exists a family of proper and connected algebraic curves $g : U \to V$ whose geometric fibers irreducible rational curves with cycle morphism $u : U \to X$, such that:*

$$u^{(2)} \colon U \times_V U \to X \times X$$

is dominant.

Working in arbitrary characteristic we need to consider the notion of separably rationally connected varieties. We recall that a morphism of varieties $f \colon X \to Y$ is *separable* if it is dominant and the field extension $K(X) \supset K(Y)$ is separably generated.

Definition 4. *A variety X is* separably rationally connected *if there is a variety V and a morphism $u \colon \mathbb{P}^1 \times V \to X$ such that*

$$u^{(2)} \colon \mathbb{P}^1 \times \mathbb{P}^1 \times V \to X \times X$$

is separable, or equivalently, is dominant and smooth at the generic point (cf. [22, Definition IV.3.2]).

If the ground field has characteristic zero, then Definition 3 and Definition 4 are equivalent, as shown in the following Proposition.

Proposition 4. *[22, Proposition IV.3.3.1] If X is separably rationally connected, then X is rationally connected, and the converse holds if the ground field has characteristic zero.*

The following property, that is a direct consequence of the definition, shows that separably rationally connectedness and rationally connectedness are birational invariant.

Proposition 5. *[22, Proposition IV.3.3.1] If X and X' are two proper varieties that are birationally equivalent, then X is (separably) rationally connected if and only if X' is.*

It is trivial to see that the following chain of implications holds:

$$\text{Rational} \Longrightarrow \text{Unirational} \Longrightarrow \text{Rationally Connected}.$$

The implications cannot be reversed when $\dim X \geq 3$ (cfr. [7] and [20]). It is unknown an example of a variety that is rationally connected but not unirational.

The notion of rational connected varieties was introduced by Campana ([6]) and Kollar-Miyaoka-Mori ([24]). Beside their easy and natural definition, rational and unirational varieties of higher dimension are hard to study and their full understanding is widely open. Rational connected varieties seem to be the right class of variety to consider. Indeed, the class of rational connected varieties contains rational and unirational varieties and rational connected varieties can be studied through the deformation of rational curves.

We recall some general facts about deformation of rational curves. For more details we refer to [22].

In general, when we have two varieties X and Y and a morphism $f : Y \to X$, to study the deformations of f is useful to consider a space that parametrizes all the morphism from Y to Y. It is known that when X is quasi-projective and Y is projective there exists a scheme $\text{Hom}(Y, X)$ that parametrized all the morphism from Y to X (cf. [22, I.1.10]). The scheme $\text{Hom}(Y, X)$ can have countable many irreducible components. However every irreducible component is of finite type over k.

Since we are interested in deformation of curves, let C be a curve and $\text{Hom}(C, X)$ be the scheme that parametrized all the morphism from C to X. We denote with $[f]$ the point of $\mathrm{Hom}(C,X)$ corresponding to the morphism $f : C \to X$. Some interesting properties of $\mathrm{Hom}(C,X)$ can be summarized in the following Proposition (cf. [22, Theorem I.2.16] and [22, Theorem II.1.3]).

Proposition 6. *Let X be a locally complete intersection variety of pure dimension n, and C be a smooth curve. Let $f : C \to X$ be a morphism. Assume that every irreducible component of C intersects the smooth locus of X. Then:*

1. $\mathrm{T}_{[f]}\mathrm{Hom}(C,X) \cong H^0(C, f^*\mathrm{T}X)$.
2. $\dim_{[f]} \mathrm{Hom}(C,X) \geq -K_X \cdot C + n\chi(\mathcal{O}_C)$

If X is smooth then we have the inequality

$$\dim_{[f]} \mathrm{Hom}(C,X) \geq h^0(C, f^*\mathrm{T}X) - h^1(C, f^*\mathrm{T}X) = -K_X \cdot C + n\chi(\mathcal{O}_C),$$

and the equality holds if $h^1(C, f^*\mathrm{T}X) = 0$. In particular, when $h^1(C, f^*\mathrm{T}X) = 0$, the space $\mathrm{Hom}(C,X)$ has a simple description near the point $[f]$.

We recall that every vector bundle over $\mathbb{P}^1$ can be decomposed as a direct sum of line bundles (cf. [14, V Exercise 2.6]). In particular, when $C \cong \mathbb{P}^1$, the pull-back on C of the tangent bundle of X can be written as

$$f^*\mathrm{T}X \cong \sum_{i=1}^{\dim X} \mathcal{O}_{\mathbb{P}^1}(a_i)$$

for suitable integers a_i.

Definition 5. *A morphism $f : \mathbb{P}^1 \to X$ is said* free *(*very free*), if all the integers a_i are ≥ 0 (> 0).*

We have the following characterization of separably rationally connected varieties (cf. [22, Theorem IV.3.7]).

Theorem 3. *Let X be a smooth variety over an algebraic closed field. Then X is separably rationally connected if there exists a very free morphism $f : \mathbb{P}^1 \to X$.*

It is straightforward from the definitions that a proper rational variety is separably rationally connected, and the converse holds in dimension two (cf. [22, Exercise IV.3.3.5]).

Proposition 7. *Let X be a proper surface. If X is separably rationally connected, then it is rational.*

Proof. By [26], there exists a resolution of singularities of X. Since both rationality and separably rational connectedness are birational properties, we may thus assume without loss of generality that X is smooth. Then, by [22, Theorem IV.3.7], there is a morphism $g\colon \mathbb{P}^1 \to X$ such that f^*T_X is ample. This implies that every section of $(\wedge^q \Omega_X)^{\otimes m}$, for any $q, m \geq 1$, vanishes along $g(\mathbb{P}^1)$. As these curves cover a dense set in X, we conclude that all sections of $(\wedge^q \Omega_X)^{\otimes m}$ are zero. Therefore X is rational by Castelnuovo's criterion. □

2.2 Factorization of birational maps

In this section we give an overview of what is known regarding factorization of birational maps.

Let X and Y be two varieties of dimension n defined over $\mathbb{C}$. We recall that a *rational* map $f : X \dashrightarrow Y$ is an equivalence class of pairs (f_U, U) in which f_U is a morphism of varieties from an open set $U \subset X$ to Y, and two such pairs (f_U, U) and (f'_U, U') are considered equivalent if f_U and f'_U coincide on the intersection $U \cap U'$. A rational map $f : X \dashrightarrow Y$ is said to be birational if it admits an inverse.

The problem is to understand if there exists a good factorization of f. The situation is clear in dimension two, as we will discuss later. The question in higher dimension is contained in the following long-standing conjecture.

Conjecture 1. *Let X and Y be two smooth varieties defined over an algebraic closed field k of characteristic zero. Let $f : X \dashrightarrow Y$ be a birational map and $U \subset X$ be an open set where f is an isomorphism. Then f factors as follows*

$$\begin{tikzcd} & Z \arrow[dl, "g"'] \arrow[dr, "h"] & \\ X \arrow[rr, dashed, "f"] & & Y \end{tikzcd}$$

where g and h are compositions of blow-ups along smooth centers disjoint from U.

A factorization as in Conjecture 1 is called *strong factorization.*

The situation in dimension two is completely clear and Conjecture 1 is true due to the following Theorem due to Zariski ([32]).

Theorem 4. *Let $f : X \dashrightarrow Y$ be a birational map between smooth complex surfaces and $U \subset X$ be the open set where f is an isomorphism. Then f factors as follows*

$$\begin{tikzcd} & Z \arrow[dl, "g"'] \arrow[dr, "h"] & \\ X \arrow[rr, dashed] & & Y \end{tikzcd}$$

where g and h are compositions of blow-ups at points disjoint from U.

In arbitrary dimension Conjecture 1 is widely open. The best result known is the Weak Factorization Theorem ([1],[33]), which states that a birational morphism can always be factored in a sequence of smooth blow-ups and blow-downs along smooth centers. We recall the precise statement (cf. [1, Theorem 0.0.1]).

Theorem 5. *Let X and Y be smooth complete varieties defined over an algebraically closed field k of characteristic zero. Let $f : X \dashrightarrow Y$ be a birational map and $U \subset X$ be an open set where f is an isomorphism. Then f can be factored into a sequence of blow-ups up and blow-downs with smooth irreducible centers disjoint from U, namely, there exists a sequence of birational maps between smooth complete varieties*

$$X = Z_0 \overset{g_0}{\dashrightarrow} Z_1 \overset{g_1}{\dashrightarrow} \cdots\cdots \overset{g_{k-2}}{\dashrightarrow} Z_{k-1} \overset{g_{k-1}}{\dashrightarrow} Z_k = Y$$

where

- $f = g_{k-1} \circ g_{k-2} \circ \dots \circ g_1 \circ g_0$*;*
- *all the g_i are isomorphism on U;*
- *either $g_i : Z_i \dashrightarrow Z_{i+1}$ or $g_i^{-1} : Z_{i+1} \dashrightarrow Z_i$ is a blow-up along a smooth center.*

Furthermore, there is an index i_0 such that for all $i \leq i_0$ the map $Z_i \dashrightarrow X$ is a projective morphism, and for all $i \geq i_0$ the map $Z_i \dashrightarrow Y$ is a projective morphism.

2.3 Varieties of minimal degree

In this section we recall some results that will be useful in Chapter 3. For more details we refer to [13].

A variety $X \subset \mathbb{P}^N$ of dimension n is said to be *nondegenerate* if it is not contained in a hyperplane. Given a nondegenerate variety $X \subset \mathbb{P}^N$, we have the following lower bound.

Proposition 8. *If $X \subset \mathbb{P}^N$ is a nondegenerate variety, then* $\deg X \geq 1 + \mathrm{codim} X$.

An interesting class of nondegenerate varieties are the varieties of minimal degree.

Definition 6. *A variety $X \subset \mathbb{P}^N$ is called* variety of minimal degree *if X is nondegenerate and* $\deg X = 1 + \mathrm{codim} X$.

The problem of classifying varieties of minimal degree is classical and it goes back to the works of Del Pezzo [8] and Bertini [5]. Clearly, the case of codimension 1 is trivial. For the general case we have the following.

Theorem 6. [Theorem 1,[13]] *If $X \subset \mathbb{P}^N$ is a variety of minimal degree, then X is a cone over a smooth such variety. If X is smooth and* $\mathrm{codim} X > 1$, *then $X \subset \mathbb{P}^N$ is either a rational normal scroll or the Veronese surface $\mathbb{P}^2 \subset \mathbb{P}^5$.*

We recall that given a variety $X \subset \mathbb{P}^r$ and a linear subspace $L \subset \mathbb{P}^{r+s+1}$ of dimension s, the *cone over X*, denoted $C(X)$, is the closure of $p_L^{-1}(X)$, where $p_L : \mathbb{P}^{r+s+1} \to \mathbb{P}^s$ is the projection from L.

A *rational normal scroll* is a cone over a smooth linearly normal variety fibered over $\mathbb{P}^1$ by linear spaces.

We give the following definition that characterizes the rational normal scrolls.

Definition 7. *Let $0 \leq a_0 \leq a_1, ... \leq a_d$ be d non negative integers, with $a_d > 0$. Let*

$$\phi_i : \mathbb{P}^1 \to \mathbb{P}^{a_i} \subset \mathbb{P}^{\sum a_i + d}$$

be the parametrized rational normal curve of degree a_i. The variety $S(a_0, a_1, ..., a_d) \subset \mathbb{P}^{\sum a_i + d}$ is the union over $\lambda \subset \mathbb{P}^1$ of the d-plane spanned by $\phi_0(\lambda), \phi_1(\lambda), ..., \phi_d(\lambda)$.

It turns out that every rational normal scroll is a variety $S(a_0, a_1, ..., a_d)$ for suitable integers $a_0, a_1, ..., a_d$.

We conclude stating the following Corollary that follows from the Theorem 6 and whose proof is elementary. We will refer to the Corollary in the third chapter.

Corollary 2. *Let $X \subset \mathbb{P}^N$ be a variety of minimal degree. If* $\dim X = 3$ *and* $\mathrm{codim} X \leq 3$, *then X is one of the following:*

- $Q^3 \subset \mathbb{P}^4$*;*
- $S(0, 0, 2)$*;*
- $S(0, 1, 1)$*;*
- $S(0, 0, 3)$*;*

- $S(1,1,1)$;
- $S(0,1,2)$;
- $S(0,0,4)$;
- $S(0,1,3)$;
- $S(1,1,2)$;
- $S(0,2,2)$;
- $C(v_2(\mathbb{P}^2)) \subset \mathbb{P}^6$.

CHAPTER 3

FAMILIES OF RATIONAL VARIETIES

In this chapter we study how rationality deforms in families. In the first section we give a set of definitions and we discuss some general properties. In the second section we study the particular case of families of rational threefolds. The third section is left for further remarks.

3.1 General properties

Let $f\colon X \to T$ be a projective equidimensional morphism onto a connected reduced scheme T of finite type over an algebraically closed field k. Assume that the fibers $X_t := f^{-1}(t)$ are varieties for all $t \in T$, and let n denote the relative dimension of f. We will refer to f as a family of projective varieties. We are interested in understanding the algebraic structure of the *rational locus*

$$\mathrm{Rat}(f) := \{\, t \in T \mid X_t \text{ is a rational variety} \,\}$$

of the family.

It follows by general facts that Rat(f) is a countable union of locally closed subsets of T (see Proposition). Once singularities are allowed, it is easy to pick up examples of families of rational varieties that specialize to nonrational ones, e.g., a family of smooth cubic surfaces in $\mathbb{P}^3$ that degenerates to a cone over an elliptic curve.

In characteristic zero, however, the following question regarding smooth families has been around for some time.

Question 3. *Assuming that $f\colon X \to T$ is a smooth family of projective varieties over an algebraically closed field of characteristic zero, is* Rat(f) *equal to a countable union of closed subsets of T?*

The answer is trivial in dimension one, and follows from Castelnuovo's rationality criterion in dimension two. It is expected that in higher dimensions $\mathrm{Rat}(f)$ can be a proper subset, possibly with infinitely many components; this should occur for instance in smooth families of cubic fourfolds (see Example 1).

Since, we do not put conditions on the characteristic of the ground field k, we consider the *separably rationally connected locus*

$$\mathrm{SRC}(f) := \{\, t \in T \mid X_t \text{ is separably rationally connected} \,\}$$

of the family.

Regarding the general structure of $\mathrm{SRC}(f)$, several interesting cases are covered by the following proposition.

Proposition 9. *Let $f \colon X \to T$ as above.*

1. *In any setting where resolution of singularities exists,* $\mathrm{SRC}(f)$ *is a constructible subset of T.*
2. *If f is smooth, then* $\mathrm{SRC}(f)$ *is open in T.*
3. *If f is smooth and k has characteristic zero, then* $\mathrm{SRC}(f)$ *is open and closed in T (and thus is either empty or equal to T).*

Proof. The assertions in (b) and (c) are proven in [22, Theorem IV.3.11]. Regarding (a), first note that f is separable as it has reduced fibers (cf. [14, Theorem II.8.6A and Proposition II.8.10]) , and so is the restriction of f over any locally closed subset of T. Let $Y \to X$ be a resolution of singularities, and consider the composition map $g : Y \to T$. Since g is separable, there is a nonempty open set T° in the regular locus of T over which the induced map $g^\circ \colon g^{-1}(T^\circ) \to T^\circ$ is smooth (the proof of [14, Corollary III.10.7] goes through without assumptions on the characteristic of the ground field as long as one assumes that the morphism is separable). By (b), $\mathrm{SRC}(g^\circ)$ is an open subset of T°. Note on the other hand that $\mathrm{SRC}(f) \cap T^\circ = \mathrm{SRC}(g^\circ)$ by Proposition 5, since every fiber of g° is birational to the corresponding fiber of f. Thus the assertion follows by Noetherian property, by considering a suitable stratification of T. □

The general structure of $\mathrm{Rat}(f)$ is described in the following property.

Proposition 10. $\mathrm{Rat}(f)$ *is a countable union of locally closed subsets of T.*

We learned the following proof, which simplifies our original arguments, from Claire Voisin.

Proof. Let $P := T \times \mathbb{P}^n$, where n is the relative dimension of f. First observe that every closed subscheme $Z \subset X \times_T P$ determines a birational map $X_t \to P_t \cong \mathbb{P}^n$ for every t such that Z_t is irreducible and both projections $Z_t \to X_t$ and $Z_t \to P_t$ are birational; conversely, all birational maps from fibers of f to $\mathbb{P}^n$ arise in this way.

Let H be an irreducible component of the relative Hilbert scheme $\mathrm{Hilb}(X \times_T P/T)$ of $X \times_T P$ over T, and let $U \to H$ be the universal family: U is a closed subscheme of $X \times_T P \times_T H$, flat over H. Consider the projections $p\colon U \to X \times_T H$ and $q\colon U \to P \times_T H$. The set

$$\{ h \in H \mid U_h \text{ is irreducible and } p_h\colon U_h \to X \text{ and } q_h\colon U_h \to P \text{ are birational} \}$$

is constructible in H. By Chevalley's theorem, its image in T is also constructible, and as such can be written as a finite union of locally closed subsets. The union of all these sets, as H varies among the irreducible components of the Hilbert scheme, is $\mathrm{Rat}(f)$. The statement then is followed by the fact that the Hilbert scheme has countably many irreducible components. $\square$

Remark 1. *An analogous property is satisfied by the locus of unirational varieties. The argument easily adjusts to this case by relaxing the condition on $p_h\colon U_h \to X$ from being birational to being dominant.*

3.2 Families of rational threefolds

The aim of this section is to give a positive answer to Question 3 for families of rational threefolds. We prove a more general result (see Theorem 7 and Theorem 9) that gives a description of $\mathrm{Rat}(f)$ inside $\mathrm{SRC}(f)$ when the varieties in the family are defined over an algebraic closed field of arbitrary characteristic. The positive answer to Question 3 for smooth families of rational threefolds defined over a field of characteristic zero will follow as a Corollary to Theorem 7 and Theorem 9.

Theorem 7. *Let $f : X \to T$ be a projective morphism from a variety X onto a smooth curve T defined over an uncountable algebraically closed field k. Let $0 \in T$ be a closed point. Assume that X_t is a rational variety for every $t \neq 0$. Then every irreducible component D of X_0 that is separably rationally connected is rational.*

Proof. Since we are assuming that k is uncountable, the argument in the proof of Proposition 10 implies that there exists a closed subscheme $Z \subset X \times_T (T \times \mathbb{P}^3)$ such that Z_t

is irreducible and both projections $Z_t \to X_t$ and $Z_t \to \{t\} \times \mathbb{P}^3$ are birational for every t in a nonempty open subset of T. This gives a birational map $f : X \to T \times \mathbb{P}^3$ over T. Let D be an irreducible component of X_0, and assume that D is separably rationally connected. Since D is a prime divisor on X, the vanishing order at its generic point defines a divisorial valuation on the function field of X. Let ν be the induced valuation on the function field of $T \times \mathbb{P}^3$. The commutativity of the diagram

$$\begin{tikzcd} X \arrow[rr, dashed, "f"] \arrow[dr, "f"'] & & T \times \mathbb{P}^3 \arrow[dl] \\ & T & \end{tikzcd}$$

implies that the center C_0 of ν in $T \times \mathbb{P}^3$ is contained in the fiber $\{0\} \times \mathbb{P}^3$.

Consider the sequence of blow-ups

$$\cdots \quad \to Y_i \to Y_{i-1} \to \quad \cdots \quad \to Y_1 \to Y_0 := T \times \mathbb{P}^3$$

where each $g_i \colon Y_i \to Y_{i-1}$ is the blow-up of Y_{i-1} along the center C_{i-1} of ν. Note that, for every i, C_i is contained in the exceptional divisor of the blow-up g_i, and $g_i(C_i) = C_{i-1}$.

By induction on i, both Y_{i-1} and C_{i-1} are smooth at the generic point of C_{i-1}, and therefore there is a dense open set $Y_{i-1}^\circ \subset Y_{i-1}$, contained in the regular locus of Y_{i-1}, such that $C_{i-1}^\circ := C_{i-1} \cap Y_{i-1}^\circ$ is smooth and the induced map $g_i^{-1}(Y_{i-1}^\circ) \to Y_{i-1}^\circ$ is the blow-up of the normal bundle $\mathcal{N}_{i-1}$ of C_{i-1}° in Y_{i-1}°. In particular, the restriction of the exceptional locus of g_i over C_{i-1}° is isomorphic to the projective bundle $\mathbb{P}_{C_{i-1}^\circ}(\mathcal{N}_{i-1})$.

It follows by a theorem of Zariski (cf. [23, Lemma 2.45]) that there is an integer $m \geq 0$ such that the center C_m of ν has codimension one in Y_m. We can pick m to be the least integer with this property. Note that C_m is equal to the proper transform of D under the birational map $X \dashrightarrow Y_m$.

If $m = 0$, then the center of ν in $T \times \mathbb{P}^3$ is the whole fiber $\{0\} \times \mathbb{P}^3$. This means that ϕ induces a birational map from D to $\{0\} \times \mathbb{P}^3$, and therefore D is rational.

Suppose then that $m \geq 1$. In this case the projection $C_m \to C_{m-1}$ is a surjective map from a threefold to a variety of dimension at most two. Note that C_m is separably rationally connected, since it is birational to X_0 which is separably rationally connected by hypothesis, and being separably rationally connected is a birational property (see Proposition 5). Since the map $C_m \to C_{m-1}$ is smooth over C_{m-1}°, it follows that C_{m-1} is separably rationally connected too. The assumption on the relative dimension of f implies that $\dim C_{m-1} \leq 2$. If C_{m-1} has dimension at most one then it is clearly rational, and the same conclusion holds if C_{m-1} is a surface by Proposition 7. Note, on the other hand, that C_m contains $g^{-1}(C^\circ_{m-1})$ as a dense open set, and the latter is isomorphic to $\mathbb{P}_{C^\circ_{m-1}}(\mathcal{N}_{m-1})$. We conclude that C_m is rational. Therefore D is rational. □

If the ground field k has characteristic zero then one can prove the analogous result using an alternative argument, based on the Weak Factorization Theorem [1, 33].

Theorem 8. *Let $f : X \to T$ be a projective morphism from a variety X onto a smooth curve T defined over an uncountable algebraically closed field k of characteristic zero. Let $0 \in T$ be a closed point. Assume that X_t is a rational variety for every $t \neq 0$. Then every irreducible component D of X_0 that is separably rationally connected is rational.*

Proof. Let $\phi\colon X \dashrightarrow T \times \mathbb{P}^n$ be as in the proof of the theorem 7, and suppose that ϕ contracts the divisor D (so that it does not induce directly a birational map from D to $\{0\} \times \mathbb{P}^3$). Let $Y \to X$ be a resolution of singularities. By the Weak Factorization Theorem applied to the induced birational map $Y \dashrightarrow T \times \mathbb{P}^n$, we can find a sequence of blow-ups p_i and blow-downs q_j with smooth irreducible centers

$$\begin{tikzcd}[column sep=small] & Z^1 \arrow[dl, "p_1"'] \arrow[dr, "q_1"] & & Z^2 \arrow[dl, "p_2"'] \arrow[dr, "q_2"] & & & & Z^k \arrow[dl, "p_k"'] \arrow[dr, "q_k"] & \\ Y = Y^0 & & Y^1 & & Y^2 & \cdots & Y^{k-1} & & Y^k = T \times \mathbb{P}^3 \end{tikzcd}$$

(we allow isomorphisms among the maps p_i and q_j). Since ϕ contracts D, there is a model Z^i, for some $1 \le i \le k$, where the proper transform D^i of D is the exceptional divisor of $q_i\colon Z^i \to Y^i$. Since D^i is rationally connected, so is its image $W_i := q_i(D^i)$, which is therefore rational. This implies that D^i is rational, since it is isomorphic to the projectivization of the normal bundle of W_i in Y^i. Therefore D is rational. □

Theorem 9. *For every family $f\colon X \to T$ of projective varieties of dimension three over an algebraically closed field,* Rat(f) *is a countable union of closed subsets of* SRC(f)*.*

Proof. The statement of the theorem is trivial if k is finite or countable, since in this case any subset of T can be expressed as a countable union of closed subsets. Thus we can assume that k is uncountable.

By Proposition 10, $\mathrm{Rat}(f)$ is a countable union of locally closed subsets of $R_i \subset T$. Suppose that $\mathrm{Rat}(f)$ cannot be written as a countable union of closed subsets of $\mathrm{SRC}(f)$. Then we can find a point $p \in \mathrm{SRC}(f) \smallsetminus \mathrm{Rat}(f)$ that belongs to the closure $\overline{R}_i$ of R_i in T

for some i. Let $S \subset \overline{R}_i$ be a curve passing through p and with generic point in R_i. Let $S' \to S$ be the normalization of S and fix a point $0 \in S'$ in the preimage of p. Let then $T' \subset S'$ be an open neighborhood of 0 such that $T' \smallsetminus \{0\}$ maps into R_i. By taking the base change

$$f' \colon X' := X \times_T T' \to T',$$

we reduce to the setting of Theorem 7, which implies that X'_0 is rational. Since $X'_0 \cong X_p$, this contradicts the fact that $p \notin \mathrm{Rat}(f)$. □

Corollary 3. *For a smooth family $f \colon X \to T$ of projective threefolds over an algebraically closed field of characteristic zero,* Rat(f) *is a countable union of closed subsets of T.*

Proof. In the hypothesis of Conjecture 3, assume that f has relative dimension 3. Suppose that $\mathrm{Rat}(f) \neq \emptyset$. Then $\mathrm{SRC}(f)$ is nonempty, and thus it is equal to T by Proposition 9. Therefore the corollary reduces to a special case of Theorem 7. □

3.3 Examples and Remarks

It is easy to construct examples of families of rational projective varieties degenerating to singular varieties that are not rational, and vice versa. We do not know any example of a (connected) smooth family of projective varieties containing both rational and non-rational members. It is expected in general that one needs to consider countable unions in Proposition 10 and in Question 3. We describe a good candidate in the example that follows.

Example 1. *Complex cubic fourfolds in $\mathbb{P}^5$ form a particularly interesting class of varieties from the point of view of rationality. The quest for rational examples goes back at least to Morin [27], who gave an incorrect argument that would have implied that the general cubic in $X \subset \mathbb{P}^5$ is rational. A codimension one family of smooth rational cubics fourfolds was detected and described in several ways by Fano [9], Tregub [29], and Beauville and Donagi [3]. A crucial step in the study of cubic fourfolds is Voisin's proof of a Torelli Theorem for these varieties [30]. More examples of rational cubic fourfolds were found by Zarhin [31], and later Hassett [16, 15] constructed a countable series of distinct families of smooth rational cubic fourfolds: these are parameterized by divisors on the family of cubics containing a plane, which has codimension one in the whole space of cubics. It is conjectured on the other hand that not only the general cubic in $\mathbb{P}^5$, but also the very general element among those containing a plane is not rational. The conjecture* *is implicit in Hassett's work: it can be formulated in terms of the transcendental lattice in* $H^4(X,\mathbb{Z})$ *or the Brauer–Manin obstruction (cf. [17]), and has eventually been formalized in the language of derived categories by Kuznetsov [25]. Knowing this conjecture would give an example of a family where the rational locus is, strictly speaking, a countable union of closed subfamilies.*

We conclude with the following Remark.

Remark 2. *Although this is merely speculation, it is possible that Question 3 might have some impact towards the understanding of the rationality problem for cubic fourfolds. It was proven by Clemens and Griffiths [7] that smooth cubic threefolds* $V \subset \mathbb{P}^4$ *are nonrational. Even though they are expected, at least by some mathematicians, to be* stably rational *(which would mean that* $V \times \mathbb{P}^k$ *is rational for some* k*), it is conceivable that* $V \times \mathbb{P}^1$ *may still be nonrational. If this were the case, then a positive answer to Question 3 would imply that the general cubic fourfold is not rational (in any characteristics). This could be seen indeed by simply taking a degeneration to a cone over* V*, since such a cone is easily seen to be separably rationally connected.*

CHAPTER 4

RATIONAL DEGREE

In this chapter we introduce the notion of rational degree. In the first section we explain the point of view that was inspired by an observation of Ionescu and Russo [19]. In the second section we give the definition of rational degree and we classify rational varieties for small values of the rational degree. In the third section we give several examples to show that all the cases of the classification's results obtained in section 2 are effective.

In this chapter the ground field is the field of the complex numbers. Given a projective variety X, we denote with $\mathrm{Chow}_1^{rat}(X)$ the Chow variety parametrizing 1-cycle with rational components.

4.1 A criterion for rationality

In [19] it was observed that one can characterize rationality in terms of suitable families of rational curves. The idea is that if there exists a birational map $\phi : \mathbb{P}^n \dashrightarrow X$, then one can look at the family of rational curves through a general point of $x \in X$ induced by the lines through a general point $p \in \mathbb{P}^n$.

For the convenience of the reader we restate and prove the result.

Theorem 10 (Ionescu-Russo)**.** *Let X be a projective variety of dimension n. Then X is rational if and only if for some $x \in X_{reg}$ there exists a closed subscheme $V \subset \mathrm{Chow}_1^{rat}(X)_x$ such that:*

i) if $U \to V$ is the universal family over V, then the tautological morphism $U \to X$ is surjective.

ii) all the curves parametrized by V are smooth at the point x.

iii) the general curve parametrized by V is uniquely determined by its tangent vector at x.

Proof. The only if part is clear. Choosing a birational map $X \dashrightarrow \mathbb{P}^n$, consider the family of rational curves on X induced by the lines through a general point of $\mathbb{P}^n$.

The if part. Suppose that for some $x \in X_{reg}$, there exists a family of rational curves parametrized by a closed subscheme $V \subset \mathrm{Chow}_1^{rat}(X)_x$ as in the statement. Let $u : U \to V$ be the universal family and $\tau : U \to X$ be the tautological morphism. By assumptions τ is surjective and u admits a section $\mathcal{E}$, which is contracted by τ to x. Let $\mathrm{Bl}_x(X)$ be the blow-up of X at the point $x = \tau(\mathcal{E})$, with exceptional divisor $E \cong \mathbb{P}^{n-1}$. Due to the smoothness assumption, the morphism τ lifts to a morphism $\widetilde{\tau} : U \longrightarrow \mathrm{Bl}_x(X)$. Let $\widetilde{\tau}_{\mathcal{E}} : \mathcal{E} \longrightarrow E \cong \mathbb{P}^{n-1}$ be the restriction of $\widetilde{\tau}$ to $\mathcal{E}$, in particular a morphism $\nu : V \to E$ is well defined. The following picture summarizes the situation:

$$\begin{tikzcd}
\mathcal{E} \arrow[r, hook] \arrow[rrr, bend left, "\widetilde{\tau}_{\mathcal{E}}"] & U \arrow[r, "\widetilde{\tau}"] \arrow[d, "u"'] \arrow[dr, "\tau"] & \mathrm{Bl}_x(X) \arrow[d] & E \cong \mathbb{P}^{n-1} \arrow[l, hook'] \\
& V \arrow[urr, bend right, "\nu"'] & X &
\end{tikzcd} \tag{4.1}$$

By the surjectivity of the map τ and by condition *iii*) of the statement, it follows that the morphism $\widetilde{\tau}_{\mathcal{E}}$ is birational. In particular, $\mathcal{E}$ is rational.
Now consider the normalization $\widetilde{U}$ of U. This is generically a $\mathbb{P}^1$-bundle over the rational variety V with a section. This shows that the variety U is rational.
To conclude the proof we have to show that the morphism $\widetilde{\tau} : U \longrightarrow \mathrm{Bl}_x(X)$ is generically one to one. To this end, consider the tangent space $T_v(U) = T_v(\mathcal{E}) \oplus (\mathcal{N}_{\mathcal{E}|U})_{|v}$ at a general point $v \in \mathcal{E}$. Since the induced map $\mathrm{d}\widetilde{\tau} : T_v(U) \longrightarrow T_{\widetilde{\tau}(v)}\mathrm{Bl}_x(X)$ splits in the two invertible linear maps $T_v(\mathcal{E}) \to T_{\widetilde{\tau}(v)}(\mathrm{Bl}_x(X))$ and $(\mathcal{N}_{\mathcal{E}|U})_{|v} \to T_{\widetilde{\tau}(v)}(\mathrm{Bl}_x(X))$, whose images in $T_{\widetilde{\tau}(v)}(\mathrm{Bl}_x(X))$ intersect only at 0, the map $\widetilde{\tau}$ does not ramify along E. This and the assumption that all the curves parametrized by C are smooth at x show that $\widetilde{\tau}$ is birational. □

Remark 3. *As pointed out by the two authors, if one weakens the condition ii) by only requiring that the general curve parametrized by V is smooth at x, then one gets a criterium for unirationality.*

Building on Ionescu-Russo's work, we make this more precise. In the Theorem that follows we show that a family of rational curves as in the Theorem 10 is actually induced by a family of lines through a point in the projective space. More, the family of rational curves parametrized by V captures the geometry of X. Indeed the universal family $U \to V$ gives a resolution of the birational map $\phi : \mathbb{P}^n \dashrightarrow X$.

Theorem 11. *Let X be a projective variety of dimension n. Suppose that X satisfies the hypothesis of Theorem 10 at some point $x \in X_{reg}$. Then there exists a birational map $\phi : \mathbb{P}^n \dashrightarrow X$ that restricts to an isomorphism in an open neighborhood of x.*

Proof. We keep the same notation as in the proof of Theorem 10. We want to construct a birational morphism $\varphi : U \to \mathbb{P}^n$, such that: the section $\mathcal{E}$ is contracted to a point $p \in \mathbb{P}^n$, and if we look at the morphism $\widetilde{\varphi}_{\mathcal{E}} : \mathcal{E} \to F \cong \mathbb{P}^{n-1} \subset \mathrm{Bl}_p(\mathbb{P}^n)$ (from $\mathcal{E}$ to the exceptional divisor F of the blown up of $\mathbb{P}^n$ at the point p, induced by the lifting $\widetilde{\varphi} : U \to \mathrm{Bl}_p(\mathbb{P}^n)$), then $\widetilde{\varphi}_{\mathcal{E}} = \widetilde{\tau}_{\mathcal{E}}$ modulo isomorphisms of E and F. So that we have the following diagram:

$$\begin{tikzcd} \mathrm{Bl}_p(\mathbb{P}^n) \arrow[d] & U \arrow[l, "\widetilde{\varphi}"'] \arrow[r, "\widetilde{\tau}"] \arrow[dl, "\varphi"] \arrow[dr, "\tau"] & \mathrm{Bl}_x(X) \arrow[d] \\ \mathbb{P}^n \arrow[rr, dashed] & & X \end{tikzcd} \tag{4.2}$$

and a well defined birational map from X to $\mathbb{P}^n$ that is an isomorphism in an open neighborhood of x.

To this end, let H be a general hyperplane in E and $D = u^*(\nu^*(H))$ be the pull back of H on U. We want to show that the morphism φ exists and it is associated to the linear system $|D + \mathcal{E}|$.

We proceed by steps.

Step 1 We want to show that $\mathcal{O}_{\mathcal{E}}(D_{\mathcal{E}} + \mathcal{E}_{\mathcal{E}}) \cong \mathcal{O}_{\mathcal{E}}$.

To this end consider the following commutative diagram:

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\widetilde{\tau}_{\mathcal{E}}"] \arrow[d, "j"] & E \arrow[d, "i"] \\ U \arrow[r, "\widetilde{\tau}"] & \mathrm{Bl}_x(X) \end{tikzcd}$$

where i and j are the natural inclusions. Observe that $\mathcal{O}_{\mathcal{E}}(\mathcal{E}_{\mathcal{E}}) \cong \mathcal{O}_{\mathcal{E}}(j^*\mathcal{E}) \cong \mathcal{O}_{\mathcal{E}}(j^*(\widetilde{\tau}^*(E))$, since the map $\widetilde{\tau}$ is generically one to one and the rational curves parametrized by V are all smooth at the point x. By the commutativity of the diagram we get that $\mathcal{O}_{\mathcal{E}}(j^*(\widetilde{\tau}^*(E)) \cong \widetilde{\tau}_{\mathcal{E}}^*(i^*(E))$. Recalling that $\mathcal{O}_{\mathcal{E}}(D_{\mathcal{E}}) \cong \mathcal{O}_{\mathcal{E}}(\widetilde{\tau}_{\mathcal{E}}^*(H))$, we get that

$$\mathcal{O}_{\mathcal{E}}(D_{\mathcal{E}} + \mathcal{E}_{\mathcal{E}}) \cong \mathcal{O}_{\mathcal{E}}(\widetilde{\tau}_{\mathcal{E}}^*(H + E_E)) \cong \mathcal{O}_{\mathcal{E}}.$$

Step 2 We want to show that $H^1(U, \mathcal{O}_U(\mathcal{E})) = 0$.

Using the projection formula, we have:

$$\widetilde{\tau}_{\mathcal{E}*}(\mathcal{O}_U \otimes \widetilde{\tau}_{\mathcal{E}}^{*}\mathcal{O}_{\mathbb{P}^{n-1}}(-1)) = \widetilde{\tau}_{\mathcal{E}*}\mathcal{O}_U \otimes \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$$

and using projection formula for higher image sheaves, for $i > 0$ we get:

$$R^i(\widetilde{\tau}_{\mathcal{E}*}(\mathcal{O}_U \otimes \widetilde{\tau}_{\mathcal{E}}^{*}\mathcal{O}_{\mathbb{P}^{n-1}}(-1))) = R^i\widetilde{\tau}_{\mathcal{E}*}\mathcal{O}_U \otimes \mathcal{O}_{\mathbb{P}^{n-1}}(-1) = 0$$

since $R^i\widetilde{\tau}_{\mathcal{E}*}\mathcal{O}_U = 0$, for $i > 0$ (being E smooth, in particular only with rational singularities). Recalling that $\mathcal{O}_{\mathcal{E}}(\mathcal{O}_U \otimes \widetilde{\tau}_{\mathcal{E}}^{*}\mathcal{O}_{\mathbb{P}^{n-1}}(-1)) = \mathcal{O}_{\mathcal{E}}(-D_{\mathcal{E}}) = \mathcal{O}_{\mathcal{E}}(\mathcal{E})$, we get that $H^1(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(\mathcal{E})) = H^1(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(-1)) = 0$ (cfr Exercise 8.1, Chapter III of [14]).

If we look at the long exact sequence in cohomology induced by the following short exact sequence:

$$0 \to \mathcal{O}_U \to \mathcal{O}_U(\mathcal{E}) \to \mathcal{O}_{\mathcal{E}}(\mathcal{E}) \to 0$$

we get that $H^1(U, \mathcal{O}_U(\mathcal{E})) = 0$ (since $H^1(U, \mathcal{O}_U) = 0$ being U rational, and $H^1(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(\mathcal{E})) = 0$ from previous discussion).

Step 3 We want to show that $H^0(U, \mathcal{O}_U(D + \mathcal{E})) = n + 1$. In particular the map $H^0(U, \mathcal{O}_U(D + \mathcal{E})) \to H^0(C, \mathcal{O}_C(D + \mathcal{E}))$ is surjective, where $C \cong \mathbb{P}^1$ is the general fiber of $u : U \to V$.

Since $\mathcal{O}_U(D)$ is globally generated, by Bertini's Theorem we can choose a general section $D^1 \in |D|$ that is smooth. By construction a smooth section of $|D|$ has to be irreducible. From Step 2 and from the piece of long exact sequence induced in cohomology by the following short exact sequence:

$$0 \longrightarrow \mathcal{O}_U(\mathcal{E}) \xrightarrow{\cdot D^1} \mathcal{O}_U(D + \mathcal{E}) \longrightarrow \mathcal{O}_{D^1}(D + \mathcal{E}) \longrightarrow 0$$

we get that

$$H^0(U, \mathcal{O}_U(D + \mathcal{E})) = H^0(D^1, \mathcal{O}_{D^1}(D + \mathcal{E})) + 1.$$

Replacing U with D^1, $\mathcal{E}$ with $\mathcal{E}_{|D^1}$, $\mathrm{Bl}_x(X)$ with $\widetilde{\tau}(D^1)$ and E with $E_{\widetilde{\tau}(D^1)}$, and using the same argument as above we get that

$$H^0(D^1, \mathcal{O}_{D^1}(D + \mathcal{E})) = H^0(D^2, \mathcal{O}_{D^2}(D + \mathcal{E})) + 1,$$

where D^2 is the general (hence smooth and irreducible) section of $H^0(D^1, \mathcal{O}_{D^1}(D + \mathcal{E}))$. Proceeding by induction on the dimension, and recalling that $n - 1$ general sections of $H^0(U, \mathcal{O}_U(D))$ intersect in a general fiber C of $u : U \to V$, we get that:

$$H^0(U, \mathcal{O}_U(D + \mathcal{E})) = H^0(C, \mathcal{O}_C(D + \mathcal{E})) + n - 1 = n + 1,$$

being $C \cong \mathbb{P}^1$ and $\mathcal{O}_C(D+\mathcal{E}) \cong \mathcal{O}_{\mathbb{P}^1}(1)$.

Step 4 We want to show that the linear system $|D+\mathcal{E}|$ defines the morphism we are looking for.

Without loss of generality we can choose $< ed_1, \dots, ed_n, s >$ as set of generators for $H^0(U, \mathcal{O}_U(D+\mathcal{E}))$, where $< d_1, \dots, d_n >$ is a generating set for $H^0(U, \mathcal{O}_U(D))$, $e \in H^0(U, \mathcal{O}_U(\mathcal{E}))$ and $s \in H^0(U, \mathcal{O}_U(D+\mathcal{E}))$ is a section not vanishing on $\mathcal{E}$.
It is clear that the set of sections $< ed_1, \dots, ed_n, s >$ defines a morphism $\varphi_{|D+\mathcal{E}|} : U \to \mathbb{P}^n$ that contracts the divisor $\mathcal{E}$ to a point. Moreover there exists a lift $\widetilde{\varphi}_{|D+\mathcal{E}|} : U \to \mathrm{Bl}_p(\mathbb{P}^n)$, where $p = \varphi_{|D+\mathcal{E}|}(\mathcal{E})$. To conclude the proof we need to show that $\varphi_{|D+\mathcal{E}|}$ is generically one to one.
By Step 3, the morphism $\varphi_{|D+\mathcal{E}|}$ restricted to the general fiber C of $u : U \to \mathcal{E}$ is an isomorphism (more precisely it sends C in a line of $\mathbb{P}^n$). Moreover, if we call $\varphi_{|D|} : U \to \mathbb{P}^{n-1}$ the morphism associated to the set of sections $< d_1, \dots, d_n > \in H^0(U, \mathcal{O}_U(D))$, we have the following commutative diagram:

$$\begin{tikzcd} U \arrow[rr, "\widetilde{\varphi}_{|D+\mathcal{E}|}"] \arrow[dr, "\varphi_{|D|}"'] & & \mathrm{Bl}_p(\mathbb{P}^n) \arrow[dl] \\ & \mathbb{P}^{n-1} & \end{tikzcd}$$

i.e., the morphism $\widetilde{\varphi}_{|D+\mathcal{E}|}$ is defined over $\mathbb{P}^{n-1}$. This shows that $\varphi_{|D+\mathcal{E}|}$ is birational and it concludes the proof. $\square$

We conclude this section giving a characterization of the projective space $\mathbb{P}^n$ in terms of families of rational curves.

Proposition 11. *Let X be a smooth projective variety of dimension n. Then $X \cong \mathbb{P}^n$ if and only if there exists a family of rational curves $V \subset \mathrm{Chow}_1^{rat}(X)_x$ that satisfies the rationality criterion for some point $x \in X$ such that all the members of V are irreducible.*

Proof. The if part is trivial, if $X \cong \mathbb{P}^n$, consider the family of lines through a point $x \in \mathbb{P}^n$.

The only if part. We start proving the following:

Claim 1. *If all the curves C of the family V are irreducible then $V \cong \mathbb{P}^{n-1}$.*

Proof. The proof is based on a useful remark due to Kebekus, ([21], proof of Theorem 3.4), pointed out by Ionescu and Russo in [19]. Using the same notation as in the proof of Theorem 10, suppose that the morphism $\nu : V \to E$ contracts a curve $B \subset V$ to a point e of $E \cong \mathbb{P}^{n-1}$. Take $\overline{B}$, the normalization of B, and consider the surface S over $\overline{B}$ obtained by base change from the universal family over B. Due to the assumption, the projection $p : S \to \overline{B}$ admits a section D, the surface S is smooth along D, and the surface S is irreducible. It is well defined a morphism $f : S \to X$, via composition with the tautological morphism from U to X. The tangent morphism of f induces a morphism $f' : \mathcal{N}_{D|S} \to l_e$, where $\mathcal{N}_{D|S}$ is the geometric normal bundle of D in S and l_e is the line in the tangent space of X at x corresponding to the point $e \in E$. This tells us that the normal bundle $\mathcal{N}_{D|S}$ has to be trivial. Indeed, if $\mathcal{N}_{D|S}$ is not trivial, the map f' has a zero and the corresponding curve in the family is singular at the point x, against the assumptions. Since $\mathcal{N}_{D|S}$ is trivial the map $f : S \to X$ is a morphism over a curve B' with fiber isomorphic to D. Since all the curves parametrized by V are irreducible, then all the points of B parametrize the same 1-cycle, a contradiction. □

Hence, we have that $V \cong \mathbb{P}^{n-1}$. Without loss of generality we can replace the universal family U with its normalization. Since all the curves parametrized by V are irreducible, the universal family $u : U \to V$ is a $\mathbb{P}^1$-bundle over V. The section $\mathcal{E}$ is contracted to a smooth point by the tautological morphism $\tau : U \to X$, hence $\mathcal{O}_{\mathcal{E}}(\mathcal{E}) \cong \mathcal{O}_{\mathcal{E}}(-1)$, and $U \cong \mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{O} \oplus \mathcal{O}(-1))$. This means that U is isomorphic to the blown up of $\mathbb{P}^n$ at one point. Now consider the map $\tau : U \to X \subset \mathbb{P}^n$, where we think the variety X embedded in some projective space $\mathbb{P}^n$. The map τ is given by some linear system $\Lambda \subset H^0(U, L)$, where L is a divisor on U. Hence $L \cong \widetilde{rH} + s\mathcal{E}$, where $\widetilde{H}$ is the pull back of the hyperplane section of $\mathbb{P}^n$ via the contraction morphism $U \to \mathbb{P}^n$, and $r, s \in \mathbb{Z}$. Since $\mathcal{E}$ is contracted by τ, it follows that $L_{\mathcal{E}} \cong \mathcal{O}_{\mathcal{E}}$ and $s = 0$. Since τ is a morphism, then $r \geq 1$ and the morphism τ factors through a morphism $t : \mathbb{P}^n \to X$:

$$\begin{tikzcd} U \arrow[rr, "\tau"] \arrow[dr] & & X \subset \mathbb{P}^n \\ & \mathbb{P}^n. \arrow[ur, "t"] & \end{tikzcd}$$

Since the morphism t is generically one to one onto a smooth variety, it follows that t is actually an isomorphism. This concludes the proof. □

4.2 Rational degree

As we discussed in the previous section, if a variety X is rational, then for the general point $x \in X$ we can find a family of rational curves parametrized by a closed subscheme

$V \subset \mathrm{Chow}^1_{rat}(X)_x$ satisfying the assumption of Theorem 10. Moreover its universal family $U \to V$ carries information of the birational map $\phi : \mathbb{P}^n \dashrightarrow X$, in the sense that U is a resolution of ϕ. The situation can be summarized by Diagram 4.1:

$$\begin{tikzcd}
\mathcal{E} \arrow[r, hook] \arrow[rrr, bend left, "\widetilde{\tau}_{\mathcal{E}}"] & U \arrow[r, "\widetilde{\tau}"] \arrow[d, "u"'] \arrow[dr, "\tau"] & \mathrm{Bl}_x(X) \arrow[d] & E \cong \mathbb{P}^{n-1} \arrow[l, hook'] \\
 & V \arrow[urr, bend right, "\nu"'] & X &
\end{tikzcd}$$

and by Diagram 4.2:

$$\begin{tikzcd}
\mathrm{Bl}_p(\mathbb{P}^n) \arrow[d] & U \arrow[l, "\widetilde{\varphi}"] \arrow[r, "\widetilde{\tau}"] \arrow[dl, "\varphi"] \arrow[dr, "\tau"] & \mathrm{Bl}_x(X) \arrow[d] \\
\mathbb{P}^n \arrow[rr, dashed, "\phi"] & & X
\end{tikzcd}$$

In the above diagrams we kept the notation as in Theorem 10 and Theorem 11.

As we will discuss in the next section, we do not know if every point x on a smooth rational variety X admits an open neighborhood $x \in \mathcal{U}$ isomorphic to an open neighborhood of the projective space. In particular, it is unknown if for every point on a smooth rational variety X we can find a family of rational curves through x satisfying the assumption of Theorem 10. For this reason, we give the following definition.

Definition 8. *We say that a point x contained in the regular locus of a rational variety X* admits a global system of parameters *if there exists a family of rational curves through x satisfying the assumption of Theorem 10, or equivalently if there exists a birational map $\phi : \mathbb{P}^n \dashrightarrow X$ that restricts to an isomorphism in an open neighborhood of x.*

Looking at the Cremona transformations of the projective space $\mathbb{P}^n$ to itself, one can easily see that given a rational variety X, for the general point $x \in X_{reg}$ there exist infinitely many families of rational curves satisfying the assumption of Theorem 10. Keeping this in mind and referring to diagram 4.1 and diagram 4.2 for the notation, we define the *rational degree* of X as follows.

Definition 9. *Let X be a rational variety and $x \in X_{reg}$ be a point such that there exists a family of rational curves, parametrized by a closed subscheme $V \subset \mathrm{Chow}^1_{rat}(X)_x$, satisfying the assumption of Theorem 10. We call degree of V the integer $\tau_*(D) \cdot C$, where $D := u^*(\widetilde{\tau}^*_{\mathcal{E}}(\mathcal{O}(1)))$ and C is the general curve parametrized by V.*

Definition 10. *Let X be a rational variety.*

1) *If $x \in X_{reg}$ admits a global system of parameters, then we call rational degree at x the minimum degree among all closed subschemes $V \in \mathrm{Chow}_1^{rat}(X)_x$ satisfying the assumption of Theorem 10.*

2) *We call rational degree of X the minimum rational degree among all the points $x \in X_{reg}$ that admit a global system of parameters.*

The next Theorem applies the notion of rational degree to the well known case of smooth algebraic surfaces.

Proposition 12. *Let X be a minimal rational surface. Then the following hold:*

- *the rational degree of X is 1 if and only if $X \cong \mathbb{P}^2$;*
- *the rational degree of X is 2 if and only if $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ or $X \cong \mathbb{F}_2$;*
- *the rational degree of X is k $(k > 2)$ if and only if $X \cong \mathbb{F}_k$.*

Proof. The statement is clear when $X \cong \mathbb{P}^2$ and when $X \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Let X be a $\mathbb{F}_k$ surface with $k > 1$. The surface $\mathbb{F}_k$ can be connected to $\mathbb{P}^2$ via one link of type I and $(k-1)$ links of type II of the Sarkisov program. Looking at the family of rational curves on X induced by the lines through the general point of $\mathbb{P}^2$, we get that the rational degree of X is at most k.

It is well known that $\overline{NE}(X)$ is generated by the classes $[f]$ and $[C_0]$, where f is the general fiber and C_0 is the section with negative self intersection, say $C_0^2 = -k$. Every effective curve on X is numerically equivalent to C_0 or $aC_0 + bf$, with $a \geq 0$ and $b \geq ak$. So for every curve C on X, we have $C^2 = -a^2k + 2ab = a(-ak + 2b) \geq a(ak)$. This shows that k is a minimum. $\square$

If a smooth rational variety of arbitrary dimension has rational degree 1, then we have the following.

Proposition 13. *Let X be a smooth projective variety of dimension n. If the rational degree of X is 1, then X admits a morphism to $\mathbb{P}^n$.*

In particular, if $\rho(X) = 1$ then X is isomorphic to the projective space $\mathbb{P}^n$.

Proof. Looking at the diagram (4.1), we know that the linear system $|\tau_*(D)|$ contains $n+1$ sections A_i that define a birational map $\phi_{|\tau_*(D)|} : X \dashrightarrow \mathbb{P}^n$. It is enough to show that $\phi_{|\tau_*(D)|}$ is a morphism. If the map $\phi_{|\tau_*(D)|} : X \dashrightarrow \mathbb{P}^n$ is not a morphism, then there exists a closed subscheme $Z \subset X$ such that all the sections A_i vanish along Z. Fix a point $p \in Z$. Note that $\dim \tau^*(p) \geq 1$, since the divisors D_i, $i = 1, \ldots, n$, meet $\tau^*(p)$. For the same argument and the fact that $\tau^*(p)$ does not contain any component of the cycles parametrized by V also $\dim v_*(u_*(\tau^*(p))) \geq 1$. This shows that there exists a divisor $L \in |D|$ and a point $q \in u_*(\tau^*(p))$ such that $L \cap u^*(q) = \emptyset$. This means that $\tau_*(L) \cdot \tau_*(u^*(q)) \geq 2$, a contradiction. □

As one can see from the proof of Proposition 13, the study of rational varieties with fixed rational degree k is related to the understanding of the base locus of the birational map $\phi : X \dashrightarrow \mathbb{P}^n$, of which the total space of the family of rational curves is a resolution. When the rational degree has small value, it imposes restrictions to the base locus of the birational map.

For smooth threefolds of rational degree 2, we have the following result.

Theorem 12. *Let X be a smooth projective rational threefold with $\rho(X) \leq 2$. If the rational degree of X is 2, then X admits a morphism $X \to X_{min}$ to a variety of minimal degree X_{min}. In particular X_{min} is one of the following:*

- $Q^3 \subset \mathbb{P}^4$;
- $S(0,0,2)$;
- $S(0,0,1)$;
- $S(1,1,1)$;
- $S(0,1,2)$;
- $C(v_2(\mathbb{P}^2)) \subset \mathbb{P}^6$.

Proof. We keep the same notation as before.

Let $|\Gamma| \subset |\tau_*(D)|$ be the linear system that defines the birational map $\phi^{-1} : X \dashrightarrow \mathbb{P}^3$ (cfr. diagram (4.2)). Since the rational degree of X is two, then the indeterminacy locus Z of ϕ^{-1} can only be a point or a curve in X.

Claim 2. *All the curves parametrized by V are smooth along Z. In particular, the general divisor of the linear system $|\Gamma|$ is smooth.*

Proof. Let F' be the divisor contracted to Z by the morphism $\tau : U \to X$. Since Z is the base locus of ϕ^{-1}, it follows that the restriction map $u_{|F'} : F' \to V$ is surjective. By composition, we have a well defined morphism h from F' to $\mathbb{P}^2$:

$$h : F' \xrightarrow[u_{|F'}]{} V \xrightarrow[\nu]{} E \cong \mathbb{P}^2.$$

Suppose that a curve C is singular along Z. Let $[C] \in V$ be the point corresponding to C. Then $\nu([C]) \in \mathbb{P}^2$ is a point. We can take a line $l \subset \mathbb{P}^2$ not containing $\nu([C])$, and look at the divisor $B := u^*(\nu^*(l))$. We have that $B \in |D|$ and $B \cap u^{-1}([C]) = \emptyset$. This implies that $\tau_*(B) \cdot C > 2$, a contradiction.

By the proof of Theorem (11), we know that the general divisor of the linear system $|D|$ is irreducible and smooth. If the general divisor of $|\Gamma|$ is singular at a point $z \in Z$ then all the members of $|\Gamma|$ are singular at z. The same argument as above shows that the general divisor of $|\Gamma|$ needs to be smooth along Z. $\square$

Claim 3. *If Z is a curve, then Z is a smooth rational curve.*

Proof. Let $B \in |D|$ be a general divisor. Note that $l' := B \cap F'$ is a smooth rational curve. Since $\tau_*(D) \cdot C = 2$, it follows that $F' \cdot u^{-1}([C]) = 1$ for every curve parametrized by V. Hence the restriction morphism $h_{|l'} : l' \to \mathbb{P}^1$ is an isomorphism.

The restriction morphism $\tau_{|l'} : l' \to Z$ is surjective, being Z the base locus of $|\Gamma|$. Since the morphism τ restricted to B is birational and the image of B in X is smooth, the restriction $\tau_{|l'} : l' \to Z$ is actually an isomorphism. $\square$

Since Z is complete intersection of the general divisors $\overline{D}_1, \overline{D}_2, \overline{D}_3$, with $\overline{D}_i \in |\Gamma|$, we get that the blown-up $\mathrm{bl}_Z(X)$ of X along Z gives a resolution of the map ϕ^{-1}:

$$\begin{tikzcd} & \mathrm{Bl}_Z(X) \arrow[dl, "q"'] \arrow[dr, "p"] & \\ X \arrow[rr, dashed, "\phi^{-1}"] & & \mathbb{P}^3 \end{tikzcd} \tag{4.3}$$

Let F be the exceptional divisor of the morphism $q : \mathrm{Bl}_Z(X) \to X$. Since the rational map ϕ^{-1} maps the curves parametrized by V into lines in $\mathbb{P}^3$, it follows that the morphism p maps F to a plane $\widetilde{H} \subset \mathbb{P}^3$.

To conclude the proof we need to show that the complete linear system $|\overline{D}|$ is globally generated and $H^0(X,\overline{D}) = \overline{D}^3 + 3$, where $\overline{D} \in |\Gamma|$. Let $B \in \overline{D}$ be an effective general divisor contained in $H^0(\overline{D}, \overline{D}_{|\overline{D}})$. By the exact sequences:

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(\overline{D}) \to \mathcal{O}_{\overline{D}}(\overline{D}) \to 0 \tag{4.4}$$

$$0 \to \mathcal{O}_{\overline{D}} \to \mathcal{O}_{\overline{D}}(B) \to \mathcal{O}_B(B) \to 0 \tag{4.5}$$

we get that $h^0(X,\overline{D}) = h^0(B, B_{|B}) + 2$.

If Z is a point, one can see that $\overline{D}^3 = B^2 = 2$ and $h^0(B, B_{|B}) = 3$, being B a smooth rational curve. This shows that $h^0(X,\overline{D}) = 5 = \overline{D}^3 + 3$.

If Z is a curve, then the effective divisor $B \in H^0(\overline{D}, \overline{D}_{|\overline{D}})$ can be written as $B = C+Z$, where C is the general curve parametrized by V. The following relation holds:

$$\overline{D}^3 = B^2 = (C+Z)^2 = C^2 + 2CZ + Z^2 = 3 + Z^2. \tag{4.6}$$

Note that, if $A = p^*(H)$ is the pull back of the general plane in $\mathbb{P}^3$, then the morphism q maps A isomorphically to a divisor $\overline{D} \in |\Gamma|$, and Z is the push forward of the proper transform of a line contained in $\widetilde{H} \subset \mathbb{P}^3$. Hence, we get that $Z^2 \leq 1$.

Let g be the general fiber of $F \cong \mathbb{F}_1 \to \mathbb{P}^1$, $\widetilde{g} = p(g) \subset \widetilde{H}$ be the image in $\mathbb{P}^3$ and P the base point of the pencil determined by the $\widetilde{g}$'s. Note that $K_{\mathbb{P}^3}\widetilde{g} = -4$ and $K_{\mathrm{Bl}_Z(X)}g = -1$. By generic smoothness and by the assumption on the Picard number $\rho(X)$, the positive dimensional components of $p(\mathrm{Exc}(p))$ contained in $\widetilde{H}$ have degree at most two. This shows that $-1 \leq Z^2 \leq 1$.

By the Riemann-Roch Theorem for singular curves, we get that

$$h^0(B, B_{|B}) = (C+Z)C + (C+Z)Z + 1 + 1 - p_a(C+Z) = 2 + (1+Z^2) + 1 = 4 + Z^2.$$

This shows that $h^0(X,\overline{D}) = 6 + Z^2 = \overline{D}^3 + 3$. Moreover the linear system $|\overline{D}|$ is globally generated, being $|\overline{D}_{|\overline{D}}|$ globally generated and the map $H^0(X,\overline{D}) \to H^0(\overline{D}, \overline{D}_{|\overline{D}})$ surjective.

This shows that X admits a morphism to a variety of minimal degree X_{min}, given by the complete linear system $|\overline{D}|$.

Since $\overline{D}^3 = Z^2 + 3 \leq 4$, we have that the degree of $X_{min} \subset \mathbb{P}^{\deg X_{min}+2}$ can be at most 4 and $\mathrm{codim} X_{min} \geq 3$. Moreover, from the description above, it easy to see that the general hyperplane of $\mathbb{P}^{\deg X+2}$ cuts X_{min} along a smooth surface Y of rational degree at most two. In particular Y can be a $\mathbb{P}^1 \times \mathbb{P}^1$, a surface $\mathbb{F}_1$, a surface $\mathbb{F}_1$, or a suface isomorphic to $\mathbb{P}^2$. This tells us that X_{min} cannot be of type $S(0,0,3)$ and $S(0,0,4)$, being the general hyperplane sections isomorphic to a surface $\mathbb{F}_3$ and $\mathbb{F}_4$, respectively.

Following the list given in Corollary 2, one gets also that X_{min} cannot be of type $S(1,1,2)$, $S(0,2,2)$ and $S(0,1,3)$. Indeed, the variety $S(0,2,2) \subset \mathbb{P}^6$ is a cone over $\mathbb{P}^1 \times \mathbb{P}^1$ embedded in $\mathbb{P}^5$ by the linear system $|\mathcal{O}(2,2)|$ and the general hyperplane section of $S(0,2,2)$ is a surface $Y \cong \mathbb{P}^1 \times \mathbb{P}^1$. A variety of type $S(0,1,3)$ is a cone over a variety $S(1,3)$, and its general hyperplane section is isomorphic to a surface $\mathbb{F}_2$. Also the general hyperplane section of a variety of minimal degree of type $S(1,1,2)$ is a smooth scroll. If the linear system $|\overline{D}|$ determines a morphism to a variety of minimal degree X_{min} of degree four, then $Z^2 = 1$, that means that the general hyperplane section Y of X_{min} is isomorphic to $\mathbb{P}^2$. This concludes the proof. $\square$

We conclude the section with the following.

Corollary 4. *Let X be a smooth projective variety of dimension n and Picard number $\rho(X) = 1$. If the rational degree of X is 2 then $X \cong \mathbb{Q}^n \subset \mathbb{P}^{n+1}$.*

Proof. We use the same notation as in the Theorem. We proceed by induction on the dimension. Suppose that the statement is true for $\dim X = n-1$, with $n > 3$. Keeping the same notation as above, we want to show for $\dim X = n$.

We have a birational map $\phi^{-1} : X \dashrightarrow \mathbb{P}^n$, where the general divisor $\overline{D}$ is mapped to a plane. If we consider a family of rational curves V through a general point $x \in X$ induced by the lines through a point in $\mathbb{P}^n$, by Proposition 13 it follows that the curves of the family cannot be all irreducible. Since $\overline{D} \cdot C = 2$ for every curve C in the family V and $\rho(X) = 1$, it follows that $\overline{D}$ is the ample divisor that generates $\mathrm{Pic}(X)$.

The base locus of the linear system $|\Gamma|$ is given by a closed subscheme Z. Note that the general member of $|\Gamma|$ is smooth. Consider the general element $B \in |D|$. The morphism τ restricts to a morphism on B onto a smooth divisor on X. We have two possibilities:

i) $\dim(\tau(F \cap B)) = n-2$;

ii) $\dim(\tau(F \cap B)) < n-2$;

In case *i)* the divisor B is mapped isomorphically to a smooth rational ample divisor $\overline{D} \subset X$ with rational degree 1. By Lefschetz-Sommese theorem and Proposition 13, we have that $Y \cong \mathbb{P}^{n-1}$. By a Theorem of Bădescu (cf. [4, Theorem 5.4.10]), it follows that $X \cong \mathbb{P}^n$, a contradiction.

In case $ii)$, the divisor B is mapped to a smooth rational divisor $\overline{D} \subset X$ with rational degree 2 and $\mathrm{Pic}(\overline{D}) \cong \mathbb{Z}$. By inductive assumption $\overline{D} \cong \mathbb{Q}^{n-1} \subset \mathbb{P}^n$, and $H^0(\overline{D}, \mathcal{O}_{\overline{D}}(\overline{D})) = n+1$. Moreover, we have

$$\overline{D}^n = (\overline{D}_{|\overline{D}})^{n-1} = 2.$$

From the short exact sequence :

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(\overline{D}) \to \mathcal{O}_{\overline{D}}(\overline{D}) \to 0,$$

we deduce that $H^0(X, \mathcal{O}_X(\overline{D})) = n+2$. Since the map

$$H^0(X, \mathcal{O}_X(\overline{D})) \to H^0(\overline{D}, \mathcal{O}_{\overline{D}}(\overline{D}))$$

is surjective and $|\overline{D}_{|\overline{D}}|$ is globally generated on $\overline{D}$, we get that $|\overline{D}|$ is globally generated on X. Hence the complete linear system $|\overline{D}|$ determines a morphism from X to a variety of minimal degree $X_{min} \subset \mathbb{P}^{n+1}$ of codimension one and degree two. Since X is smooth with $\rho(X) = 1$, it is easy to see that $X_{min} \cong Q^n \subset \mathbb{P}^{n+1}$ and that the linear system $|\overline{D}|$ determines an isomorphism. The Corollary is proved. $\square$

4.3 Examples

The aim of this section is to show that all the cases of Theorem 12 are effective.

We recall, for the convenience of the reader, Diagram 4.2, which illustrates the setting in which we are working.

$$\begin{tikzcd}
\mathrm{Bl}_p(\mathbb{P}^n) \arrow[d] & U \arrow[l, "\widetilde{\varphi}"'] \arrow[r, "\widetilde{\tau}"] \arrow[dl, "\varphi"] \arrow[dr, "\tau"] & \mathrm{Bl}_x(X) \arrow[d] \\
\mathbb{P}^n \arrow[rr, dashed, "\phi"] & & X
\end{tikzcd}$$

In all the examples that follow, we call $\phi(H) = \overline{D}$ the image of the general plane $H \subset \mathbb{P}^3$, and C the rational curve induced by the general line in $\mathbb{P}^3$. We call Z the exceptional locus of τ; in all the examples that follow the exceptional divisor of τ will be the proper transform of a plane $\widetilde{H} \subset \mathbb{P}^3$. Moreover to compute $\overline{D}^3$ we use the formula 4.6:

$$\overline{D}^3 = (\overline{D}_{\overline{D}})^2 = (C+Z)^2 = C^2 + 2CZ + Z^2 = 3 + Z^2.$$

Example 2. *Let $\widetilde{H} \subset \mathbb{P}^3$ be a plane and $Q \subset \widetilde{H}$ a smooth conic. Let X be the smooth variety obtained first blowing-up $\mathbb{P}^3$ along Q, and then contracting the proper transform of $\widetilde{H}$. It is clear that $\overline{D} \cdot C = 2$ and that $X \cong Q^3 \subset \mathbb{P}^4$.*

Example 3. *Let* $\widetilde{H} \subset \mathbb{P}^3$ *be a plane and* $l \subset \widetilde{H}$ *be a line. Let* X *be the smooth variety obtained as follows: first we blow-up* $\mathbb{P}^3$ *along the line* l*, and we call* E *the exceptional divisor; then, we blow-up* $\mathrm{Bl}_l(\mathbb{P}^3)$ *along* l'*, where* l' *is the curve where* E *intersects the proper transform of* $\widetilde{H}$*; finally, we contract on* $\mathrm{Bl}_{l,l'}(\mathbb{P}^3)$ *the proper transform of* $\widetilde{H}$*. It is clear that* $\overline{D} \cdot C = 2$*. In this case* $\overline{D}^3 = 2$ *and the complete linear system* $|\overline{D}|$ *determines a morphism* $X \to X_{min} \subset \mathbb{P}^4$*, where* X_{min} *has degree two. Since the general member of* $\overline{D}$ *is mapped to a* $\mathbb{F}_2$ *surface, we conclude that* $X_{min} \cong S(0,0,2)$*.*

Example 4. *Let* $\widetilde{H} \subset \mathbb{P}^3$ *be a plane and* l_1 *and* l_2 *be two lines contained in* $\widetilde{H}$*. Let* X *be the variety obtained as follows: first, we blow-up* $\mathbb{P}^3$ *along the line* l_1*; then we blow-up* $\mathrm{Bl}_{l_1}(\mathbb{P}^3)$ *along the proper transform of* l_2*; finally, we contract the proper transform of* $\widetilde{H}$ *on* $\mathrm{Bl}_{l_1,l_2}(\mathbb{P}^3)$*. We have that* $\overline{D}^3 = 2$*, and the linear system* $|\overline{D}|$ *defines a morphism* $X \to X_{min} \subset \mathbb{P}^4$*, where* X_{min} *has degree two. Note that* $\rho(X) = 2$ *and the map* $X \to X_{min} \subset \mathbb{P}^4$ *does not contract divisors. Since the general member of* $|\overline{D}|$ *is mapped to a quadric surface* $Q \subset \mathbb{P}^3$*, it follows that* $X_{min} \cong S(0,1,1)$*.*

Example 5. *Let* X *be the smooth variety obtained first blowing-up* $\mathbb{P}^3$ *at one point* p *and along a line* l $(p \notin l)$*, and then contracting the proper transform of the plane spanned by* p *and* l *to a rational curve. One can easily that* Z *has self intersection zero on* $\overline{D}$ *and* $\overline{D}^3 = 3$*. The linear system* $|\overline{D}|$ *defines a morphism* $X \to X_{min} \subset \mathbb{P}^5$*, where* X_{min} *has degree three. One can easily see that the variety* X *is isomorphic to* $\mathbb{P}^1 \times \mathbb{P}^2$*, and the morphism* $X \to X_{min} \subset \mathbb{P}^5$ *is induced by the linear system* $|\mathcal{O}(1,1)|$*. In particular* X_{min} *is smooth and* $X_{min} \cong \mathbb{P}^1 \times \mathbb{P}^2$*. It follows that* X_{min} *is of type* $S(1,1,1)$*.*

Example 6. *Let* $\widetilde{H} \subset \mathbb{P}^3$ *be a plane, and* X *be the variety obtained as follows: first, one blows-up a line* $l \in \widetilde{H}$*; then, one blows-up* $\mathrm{Bl}_l(\mathbb{P}^3)$ *at a point* p *lying on the intersection of the proper transform of* $\widetilde{H}$ *with the exceptional divisor of* $\mathrm{Bl}_l(\mathbb{P}^3) \to \mathbb{P}^3$*; finally, ones contracts the proper transform of* $\widetilde{H}$ *on* $\mathrm{Bl}_{l,p}(\mathbb{P}^3)$ *to a smooth rational curve. One can see easily that* $\overline{D}^3 = 3$*, and the linear system* $|\overline{D}|$ *defines a morphism* $X \to X_{min} \subset \mathbb{P}^5$*. Since the general divisor* $\overline{D}$ *is mapped to a surface* $\mathbb{F}_1$*, it follows that* $X_{min} \cong S(0,1,2)$*.*

Example 7. *Let* $\widetilde{H} \subset \mathbb{P}^3$ *be a plane, and* X *be the variety obtained as follows: first, one blows-up a point* $p \in \widetilde{H}$*; then, one blows-up* $\mathrm{Bl}_p(\mathbb{P}^3)$ *along the line* l *where the proper transform of* $\widetilde{H}$ *intersects the exceptional divisor of* $\mathrm{Bl}_p(\mathbb{P}^3) \to \mathbb{P}^3$*; finally, one contracts the proper transform of* $\widetilde{H}$ *on* $\mathrm{Bl}_{p,l}(\mathbb{P}^3)$ *to a smooth rational curve. Since on* $\overline{D}$ *the curve*

Z has self intersection one, it follows that $\overline{D}^3 = 4$. The linear system $|\overline{D}|$ defines a morphism $\phi : X \to X_{min} \subset \mathbb{P}^6$. In particular, the image $\phi(\overline{D})$ of the general divisor $\overline{D}$ is isomorphic to the projective plane and $X_{min} \cong c(v_2(\mathbb{P}^2))$.

CHAPTER 5

FURTHER REMARKS

This chapter consists of two sections. In the first section we discuss the problem of the existence of global system of parameters centered at a point on a rational variety. In the second section we give a proof of Proposition 10 in chapter 2 that relies on the deformation of rational curves. In this chapter our ground field is the field of the complex numbers.

5.1 Global system of parameters

The result that we discuss in this section was inspired by the following question raised an informal way by Pandharipande:

Question 4. *If a smooth variety X of dimension n is rational, then is it the case that for every point $x \in X$ there exists an open neighborhood $\mathcal{U}$ of x isomorphic to an open set of the projective space $\mathbb{P}^n$?*

We know that if a smooth variety X of dimension n is rational then there exists an open subset $\mathcal{U} \subset X$ that is isomorphic to an open subset of the affine space $\mathbb{C}^n$. In particular, for every point in $p \in \mathcal{U}$ we can find a global system of parameters centered at p, induced by the coordinate's parameters of the affine space. It seems interesting to investigate if this property holds for every point $x \in X$.

We will say that a point x on a rational variety X *admits a global system of parameters* (shortly g.s.p) if there exists an open neighborhood $\mathcal{U}$ of x isomorphic to an open set of the projective space $\mathbb{P}^n$.

It is generally expected that the answer to Question 4 should be negative. In fact an expected counter example was the blown up of $\mathbb{P}^n$ along a subvariety far away from being rational. The points on the exceptional divisor were expected not to admit an open set isomorphic to an open set of the projective space $\mathbb{P}^n$.

We do not have a complete answer to Question 4. Our result proves that the property of admitting a global system of parameters centered at one point has a good behavior under blow up.

Lemma 1. *Let X be a projective variety of dimension n and $x \in X_{reg}$ be a point that admits a global system of parameters. If $f : \mathrm{Bl}_Z(X) \to X$ is the blow-up of X along a smooth subvariety Z containing the point x, then every point $y \in f^{-1}(x)$ admits a global system of parameters.*

Proof. Note that the proof can be reduced to prove the case when $X = \mathbb{P}^n$.

We treat separately the two cases when Z is a point and when Z has positive dimension.

The case when $f : \mathrm{Bl}_p(\mathbb{P}^n) \to \mathbb{P}^n$ is the blow-up of $\mathbb{P}^n$ at a point p is trivial. Indeed $\mathrm{Bl}_p(\mathbb{P}^n)$ admits a $\mathbb{P}^1$-bundle structure over $\mathbb{P}^{n-1}$. Since for every point $p \in \mathrm{Bl}_p(\mathbb{P}^n)$ we can find a neighborhood $p \in U \cong \mathbb{C}^{n-1} \times \mathbb{P}^1$, the statement follows easily.

Let $f : \mathrm{Bl}_Z(\mathbb{P}^n) \to \mathbb{P}^n$ be the blow-up of $\mathbb{P}^n$ along a positive dimensional smooth subvariety Z of codimension $k \geq 2$, and let F be the exceptional divisor. Fix a point $y \in F$. Let Y be a birational modification of $\mathrm{Bl}_Z(\mathbb{P}^n)$ obtained as follows.

Consider a linear subspace M of $\mathbb{P}^n$ of dimension $k-1$, such that:

i) M meets Z only at the point $f(y)$ and the intersection is transversal;

ii) the proper transform of M on $\mathrm{Bl}_Z(\mathbb{P}^n)$ does not contain the point y.

Let $N \subset M$ be a linear subspace of dimension $k-2$ that does not contain $f(y)$. By construction, the projection of Z from N to a general $\mathbb{P}^{n-k+1}$ is birational, in particular it is an isomorphism in a neighborhood of $f(y)$.

Let Y be the blow up of $\mathbb{P}^n$ along N, with exceptional divisor E. The divisor E is a $\mathbb{P}^{n-k+1}$-bundle over $N \cong \mathbb{P}^{k-2}$. Moreover the proper transform of every $(k-1)$-plane in $\mathbb{P}^n$ containing N cuts a section on E with respect to the $\mathbb{P}^{n-k+1}$-bundle structure. In particular, we have that $E \cong \mathbb{P}^{k-2} \times \mathbb{P}^{n-k+1}$ and that Y admits a $\mathbb{P}^{k-1}$-bundle structure over $\mathbb{P}^{n-k+1}$, we denote $p : Y \to \mathbb{P}^{n-k+1}$ the natural projection. Note that when p is restricted to Z, it gives the projection of Z from N to a general $(n-k+1)$-linear space.

Now, let Y_1 be the blow-up of Y along the proper transform of Z, and let F_1 be the exceptional divisor. Note that it is defined a natural birational map between Y_1 and $\mathrm{Bl}_Z(\mathbb{P}^n)$, that is an isomorphism between $Y_1 \setminus E$ and $\mathrm{Bl}_Z(\mathbb{P}^n) \setminus \overline{N}$ (where $\overline{N}$ denote the proper transform of N in $\mathrm{Bl}_Z(\mathbb{P}^n)$). Call y_1 the point on Y_1 corresponding to y via the above isomorphism. Note that to prove the statement for the point $y_1 \in Y_1$ is equivalent to prove the statement for the point $Y \in \mathrm{Bl}_Z(X)$. The variety Y_1 is the birational modification of $\mathrm{Bl}_Z(\mathbb{P}^n)$ that we work with to prove the statement.

The variety Y_1 has a $\mathbb{P}^{k-1}$-bundle structure over $\mathbb{P}^{n-k+1}$, we denote $p_1 : Y_1 \to \mathbb{P}^{n-k+1}$ the natural projection. Call E_1 the proper transform of E.

Without loss of generality, we can replace Y_1 with $p_1^{-1}(U)$, where $U \subset \mathbb{P}^{n-k+1}$ is the open subset such that for every point $u \in U$ the fiber $p_1^{-1}(u)$ either intersects F_1 transversally in a connected $(k-1)$-plane or does not meet F_1.

Set $\widetilde{Z} := p_1(Z)$ and $D := p_1^{-1}(\widetilde{Z}) - F_1$. Note that D is smooth. Moreover, the restriction of p_1 to D defines a morphism $D \to \widetilde{Z}$. Call $\widetilde{z} \in \widetilde{Z}$ the general point and $D_{\widetilde{z}}$ the general fiber of $D \to \widetilde{Z}$. Note that $D_{\widetilde{z}}$ is isomorphic to the blow-up of $\mathbb{P}^{k-1}$ at one point, and in particular admits a $\mathbb{P}^1$-bundle structure over $\mathbb{P}^{k-2}$. If l the class of the fiber of the projection $D_{\widetilde{z}} \to \mathbb{P}^{k-2}$, then that $K_{Y_1} \cdot l = -1$.

The relative Mori Cone $\overline{NE}(Y_1/U)$ is generated by classes of two curves. One class is the class of the line contained in the fiber of F_1, with respect to the $\mathbb{P}^{k-1}$-bundle structure over Z. The other class is the class l defined above. The two classes are extremal rays. The cone Theorem applies and it tells us that there exists a morphism $\rho_l : Y_1 \to Y_2$ that contracts the divisor D.

Remark 4. *Note that we can give explicitly the linear system that defines the contraction morphism $\rho_l : Y_1 \to Y_2$. Call $A_m = mH_1 - F_1 - (m-1)E_1$, where H_1 denotes the pull back of the general hyperplane of $\mathbb{P}^n$ via the morphism $Y_1 \to \mathbb{P}^n$ that contracts E_1 and F_1. Then ρ_l is given by the linear system $|A_m|$ for $m >> 0$.*

Remark 5. *The birational map $\rho_l : Y_1 \to Y_2$ is the higher dimensional analogous of the elementary transformations between two smooth surfaces $\mathbb{F}_m$ and $\mathbb{F}_{m+1}$.*

The morphism ρ_l contracts the divisor D to a $\mathbb{P}^{k-2}$-bundle over $\widetilde{Z}$.

By construction, it is defined a birational map $g : Y_1 \dashrightarrow Y_2$, in the natural way. In particular, the map g is an isomorphism in an open neighborhood of y_1. To conclude, it is enough to observe that, since Y_2 is a $\mathbb{P}^{k-2}$-bundle over U with a section, every point of Y_2 admits a global system of parameters. $\square$

We conclude this section with the following remark, in which we discuss what we expect.

Remark 6. *Although the expected answer to question 2 should be negative, it seems reasonable to expect a positive answer in stable sense. That is, if X is a smooth rational variety then every point $x \in X \times \mathbb{P}^k$ should admit a global system of parameters, where k is a suitable integer depending on the birational geometry of X.*

Due to the Weak Factorization Theorem we know that every birational map $\phi : \mathbb{P}^n \dashrightarrow X$ factors through a finite sequence of blow-ups and blow-downs along smooth centers.

$$\begin{tikzcd}[column sep=small] & Z^1 \arrow[dl, "p_1"'] \arrow[dr, "q_1"] & & Z^2 \arrow[dl, "p_2"'] \arrow[dr, "q_2"] & & & & Z^k \arrow[dl, "p_k"'] \arrow[dr, "q_k"] & \\ \mathbb{P}^n & & Y^1 & & Y^2 & \cdots & Y^{k-1} & & X \end{tikzcd}$$

Proposition 3 tells us that the first step where we may lose the property of admitting a global system of parameters at every point must be a blow-down $q_i : Z_i \to Y_i$. By replacing the blow-down $q_i : Z_i \to Y_i$ with the induced morphism $q_i' : Z_i \times \mathbb{P}^1 \to Y_i \times \mathbb{P}^1$, we gain more freedom in the degeneration of rational curves. Mori theory and degeneration of rational curve could be applied to show that the property of having a global system of parameters at every point is maintained after taking cartesian product with $\mathbb{P}^1$. Due to the finite number of steps of the factorization, one may get that the property holds for $X \times \mathbb{P}^k$, for a suitable k depending on the birational geometry of X.

5.2 Families of rational varieties through deformation of rational curves

The argument used in the proof of Theorem 9 of Chapter 2 cannot be applied to the study of deformation of rational varieties of dimension greater or equal than four. Although Proposition 10 of Chapter 2 follows by general arguments, we can prove the property through an analysis of the deformation of families of rational curves. This gives a better understanding of the geometry of the problem. Indeed, the problem could be reduced to the study of the deformation of the families of rational curves, that are proper rational varieties of dimension one less than the original varieties. A better understanding of deformation of these families of rational curves may start an inductive argument.

We restate Proposition 10 of Chapter 2 and we give the proof through the analysis of the deformation of families of rational curves. Since the proof relies on Theorem 10 of Chapter 3, for convenience of the reader we recall the Theorem.

Theorem 13. *Let X be a projective variety of dimension n. Then X is rational if and only if for some $x \in X_{reg}$ there exists a closed subscheme $V \subset \mathrm{Chow}_1^{rat}(X)_x$ such that:*

i) *if $U \to V$ is the universal family over V, then the tautological morphism $U \to X$ is surjective.*

ii) *all the curves parametrized by V are smooth at the point x.*

iii) *the general curve parametrized by V is uniquely determined by its tangent vector at x.*

Proposition 14. *Let $f\colon X \to T$ be a projective equidimensional morphism onto a connected reduced scheme T of finite type. Then* $\mathrm{Rat}(f)$ *is a countable union of locally closed subsets of T.*

Proof. The main step of the proof is to show that to every irreducible component $\mathcal{B} \subset \mathrm{Chow}^1_{rat}(X/T)$ of the Chow variety parametrizing 1-cycles with rational components of X over T corresponds a countable union of constructible subsets of T whose fibers are rational. And vice versa, if a fiber X_t, with $t \in T$, of the morphism $F : X \to T$ is rational then X_t is contained in a countable union of constructible subsets of T corresponding to some irreducible closed subscheme $\mathcal{B} \subset \mathrm{Chow}^1_{rat}(X/T)$, i.e., we want to construct the following correspondence:

$$\mathcal{B} \rightsquigarrow Z_{\mathcal{B}}$$

where $Z_{\mathcal{B}}$ is the countable union of constructible subsets of T corresponding to $\mathcal{B}$. Such that :

- for every $\mathcal{B} \subset \mathrm{Chow}^1_{rat}(X/T)$ and for every $t \in Z_{\mathcal{B}}$, the fiber X_t is rational;
- if a fiber X_t is rational then $t \in \bigcup_{\mathcal{B} \subset \mathrm{Chow}^1_{rat}(X/T)} Z_{\mathcal{B}}$.

Note that if we show this the statement easily follows. Indeed, the irreducible components of the Chow variety $\mathrm{Chow}^1_{rat}(X/T)$ are countable many and a constructible subset of T is finite disjoint union of locally closed subsets of T (see Ex 3.18 Chapter II of [14]).

For every irreducible component $\mathcal{B}$ of $\mathrm{Chow}^1_{rat}(X/T)$, let $\rho : \mathcal{U} \to \mathcal{B}$ be the universal family and let $\phi : \mathcal{U} \to X$ be the tautological morphism. We have the following diagram:

$$\begin{tikzcd}
\mathcal{U} \arrow[r, "\phi"] \arrow[d, "\rho"] & X \arrow[d, "F"] \\
\mathcal{B} & T
\end{tikzcd}$$

Suppose that $\mathcal{B}$ does not contain any closed subscheme V that satisfies the assumption of Theorem 13 at some point x in the smooth locus of some fiber X_t of F. Then to the irreducible component $\mathcal{B}$ corresponds the empty set:

$$\mathcal{B} \rightsquigarrow Z_{\mathcal{B}} = \varnothing \subset T.$$

If the irreducible component $\mathcal{B}$ contains a closed subscheme V that satisfies the assumption of Theorem 10 at some point $x \in X_t$ for some fiber X_t of F, then the fiber X_t is rational. Let $\pi : U \to V$ be the universal family. A picture of the situation is given in the following diagram:

$$\begin{tikzcd}
U \arrow[r, hook] \arrow[d, "\pi"'] \arrow[rrr, bend left] & \mathcal{U} \arrow[r, "\phi"] \arrow[d, "\rho"] & X \arrow[d, "F"] & X_t \arrow[l, hook'] \arrow[d] \\
V \arrow[r, hook] & \mathcal{B} & T & \{t\} \arrow[l, hook']
\end{tikzcd}$$

Since $F \circ \phi$ is proper, it follows that the image $F \circ \phi(\mathcal{U})$ is an irreducible closed subvariety of T, say S. If S is the only point $t \in T$, then to the irreducible component $\mathcal{B}$ corresponds just that point:

$$\mathcal{B} \rightsquigarrow Z_{\mathcal{B}} = \{t\}.$$

Suppose that S is a subvariety of T of positive dimension. We point out that in this case all the fibers of F over S are covered by rational curves. Note that the irreducible components $\mathcal{B}$ can contain one or more closed subscheme V satisfying the assumption of Theorem 13 at the fibers of F over S. This led us to consider the Chow variety $\mathrm{Chow}^{\mathrm{n-1}}(\mathcal{U}/\mathrm{T})$ parametrizing $n-1$-dimensional closed subscheme of $\mathcal{U}$ over $S \subset T$. We focus on the irreducible components of $\mathrm{Chow}^{\mathrm{n-1}}(\mathcal{U}/\mathrm{T})$ containing a class $[V]$, where V is a closed subscheme of $\mathcal{U}$ satisfying the assumption of Theorem 13 for some fiber of F. For every such irreducible component $\mathcal{D}$ of $\mathrm{Chow}^{\mathrm{n-1}}(\mathcal{U}/\mathrm{T})$, we want to define a correspondence with a constructible subset $W_{\mathcal{D}}$ of $S \subset T$, such that if $w \in W_{\mathcal{D}}$ then the fiber X_w is rational.

Fix an irreducible component $\mathcal{D}$ in $\mathrm{Chow}^{\mathrm{n-1}}(\mathcal{U}/\mathrm{T})$ containing a class $[V]$, with universal family $\nu : \mathcal{G} \to \mathcal{D}$ and tautological morphism $\eta : \mathcal{G} \to \mathcal{U}$. A picture of the situation is given in the following diagram:

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "\eta"] \arrow[d, "\nu"] & \mathcal{U} \arrow[r, "\phi"] \arrow[d, "\rho"] & X \arrow[d, "F"] \\
\mathcal{D} \arrow[rr, bend right, "h"'] & V \subset \mathcal{B} & T
\end{tikzcd}$$

Since the morphism $F \circ \phi \circ \eta : \mathcal{G} \to T$ is proper, then its image a closed subvariety W of $S \subset T$. If W is a single point $S \in S$, then we say that to the irreducible component $\mathcal{D}$ of $\mathrm{Chow^{n-1}(\mathcal{U}/T)}$ corresponds just that point:

$$\mathcal{D} \rightsquigarrow \{s\}$$

Suppose that W has positive dimension. All the subschemes parametrized by $\mathcal{D}$ are contracted to a point by the morphism $\phi \circ \eta$, hence it is well defined a morphism $h : \mathcal{D} \to W \subset S \subset T$. We have also an induced map $\delta := \pi \circ \eta : \mathcal{G} \to \mathcal{B}$. Consider the fiber product $\mathcal{G} \times_{\mathcal{B}} \mathcal{U}$ of $\mathcal{G}$ and $\mathcal{U}$ over $\mathcal{B}$, with projection map p and q respectively. Set $\widetilde{\mathcal{U}}$ the set of all points p in $\mathcal{U}$ for which the fiber over $\mathcal{B}$ passing through p is singular at p. It follows that $\widetilde{\mathcal{U}}$ is a closed subset of $\mathcal{U}$, hence $q^{-1}(\widetilde{\mathcal{U}})$ is a closed subset of $\mathcal{G} \times_{\mathcal{B}} \mathcal{U}$.

Now we look at the closed set $q^{-1}(\mathrm{Im}(\eta)) \cap \mathrm{q}^{-1}(\widetilde{\mathcal{U}})$, its image via $\nu \circ p$ is a closed subset of $\mathcal{D}$, moreover every point in its complement, say $\widetilde{R}$, parametrizes a subvariety that meets every fiber over $\mathcal{B}$ in a smooth point. Call R the set $\nu^{-1}(\widetilde{R})$, that is open. Let X' be the flat family over $\mathcal{D}$ obtained by base change of X over T. So we have:

$$\begin{tikzcd}
\mathcal{G} \arrow[rr, "\phi\circ\eta"] \arrow[dr, "\nu"'] & & X' \arrow[dl, "F'"] \\
& \mathcal{D} &
\end{tikzcd}$$

Call $\mathbb{P}_{sub}T(X')$ the projectivization of the tangent bundle of X', with projection map $\pi_T : \mathbb{P}_{sub}T(X) \to X'$. Since the general point of $\mathcal{G}$ is not contained in $\eta^{-1}(\widetilde{U})$, the morphism $\phi \circ \eta$ lifts to a rational map $g : \mathcal{G} \dashrightarrow \mathbb{P}_{sub}T(X')$:

$$\begin{tikzcd}
& \mathbb{P}T_{sub}(X) \arrow[dr, "\pi_T"] & \\
\mathcal{G} \arrow[ur, dashed, "g"] \arrow[rr, "\phi\circ\eta"] \arrow[dr] & & X' \arrow[dl, "F'"] \\
& \mathcal{D} &
\end{tikzcd}$$

Since every curve parametrized by $\mathcal{B}$ is in $\mathrm{Chow}^1_{\mathrm{rat}}(\mathrm{X/T})$, the image of $\mathcal{G}$ via g is contained in a subbundle of $\mathbb{P}_{sub}T(X')$ that at each point gives the projectivization of the tangent space of the fiber of F' containing the point. Remember that there exists a point $[V] \in \mathcal{D}$ such that the restriction $g_{|V} : V \to \mathbb{P}^{n-1} \subset \pi_T^{-1}(h([V]))$ is a morphism generically one to one. Since by construction there is no ramification, the rational map g is generically one to one. The rational map g is a morphism over an open subset of $\mathcal{G}$, in particular (by construction) over an open subset Q of $\mathcal{D}$. This tells us that the fibers of F' over the open subset Q of $\mathcal{D}$ are rational. Hence the fibers of our original flat morphism F over $h(Q)$ are rational. By Chevalley's Theorem (see Ex. 3.19 Chapter II of [14]), $h(Q)$ is a constructible subset of T. So to the irreducible component $\mathcal{D}$ corresponds the constructible subset $W_{\mathcal{D}}$ of T:

$$\mathcal{D} \rightsquigarrow W_{\mathcal{D}}$$

Since the irreducible components $\mathcal{D}$ of $\mathrm{Chow}^{n-1}(\mathcal{U}/T)$ are countable many, to the closed subscheme $\mathcal{B}$ of $\mathrm{Chow}^1_{rat}(X/T)$ corresponds the countable union of constructible subsets $W_{\mathcal{D}}$, for $\mathcal{D}$ irreducible component of $\mathrm{Chow}^{n-1}(\mathcal{U}/T)$:

$$\mathcal{B} \rightsquigarrow \bigcup_{\mathcal{D} \subset \mathrm{Chow}^{n-1}(\mathcal{U}/T)} W_{\mathcal{D}}.$$

It is easy to see that the correspondence above is the one we were looking for. This concludes the proof. $\square$

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